Herbrand universe

The Herbrand universe $D(F)$ of a closed formula F in Skolem form is the set of all terms without variables that can be constructed using the function symbols and constants of F.

In the special case that F contains no constants, we first pick an arbitrary constant, say a, and then construct the variable-free terms.

Formally, $D(F)$ is inductively defined as follows:

(1) All constants occurring in F belong to $D(F)$; if no constant occurs in F, then $a \in D(F)$.

(2) For every n-ary function symbol f occurring in F, if $t_1, t_2, \ldots, t_n \in D(F)$ then $f(t_1, t_2, \ldots, t_n) \in D(F)$.

Herbrand structure

Let F be a closed formula in Skolem form. A structure $\mathcal{A} = (U_\mathcal{A}, I_\mathcal{A})$ suitable for f is a Herbrand structure for F if it satisfies the following conditions:

(1) $U_\mathcal{A} = D(F)$, and

(2) for every n-ary function symbol f occurring in F and every $t_1, t_2, \ldots, t_n \in D(F)$: $f^\mathcal{A}(t_1, t_2, \ldots, t_n) = f(t_1, t_2, \ldots, t_n)$.

Fundamental theorem of predicate logic

Theorem: A closed formula F in Skolem form is satisfiable if and only if it has a Herbrand model.

Proof: If the formula has a Herbrand model then it is satisfiable.

For the other direction let $\mathcal{A} = (U_{\mathcal{A}}, I_{\mathcal{A}})$ be an arbitrary model of F. We define a Herbrand structure $\mathcal{B} = (U_{\mathcal{B}}, I_{\mathcal{B}})$ as follows:

Universe	$U_{\mathcal{B}} = D(F)$
Function symbols	$f^{\mathcal{B}}(t_1, t_2, \ldots, t_n) = f(t_1, t_2, \ldots, t_n)$
Predicate symbols	$(t_1, \ldots, t_n) \in P^{\mathcal{B}}$ iff $(\mathcal{A}(t_1), \ldots, \mathcal{A}(t_n)) \in P^{\mathcal{A}}$

Claim: $\mathcal{B}$ is also a model of F.

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We prove a stronger assertion:

> For every closed form G in Skolem form such that G^* only contains atomic formulas of F^*: if $\mathcal{A} \models G$ then $\mathcal{B} \models G$

Proof: By induction on the number n of universal quantifiers of G.

Basis ($n = 0$). Then G has no quantifiers at all.
It follows $\mathcal{A}(G) = \mathcal{B}(G)$, and we are done.

Step $(n > 0)$. Then $G = \forall x\ H$.

$$
\begin{array}{lll}
 & \mathcal{A} \models G & \\
\Leftrightarrow & \text{for every } u \in U_{\mathcal{A}}\text{: } \mathcal{A}_{[x/u]}(H) = 1 & \\
\Rightarrow & \text{for every } u \in U_{\mathcal{A}} \text{ of the form } u = t^{\mathcal{A}} & \\
 & \quad \text{where } t \in D(G)\text{: } \mathcal{A}_{[x/u]}(H) = 1 & \text{(Skolem!)} \\
\Leftrightarrow & \text{for every } t \in D(G)\text{: } \mathcal{A}_{[x/t^{\mathcal{A}}]}(H) = 1 & \\
\Leftrightarrow & \text{for every } t \in D(G)\text{: } \mathcal{A}(H[x/t]) = 1 & \text{(conversion lemma)} \\
\Rightarrow & \text{for every } t \in D(G)\text{: } \mathcal{B}(H[x/t]) = 1 & \text{(induction hypothesis)} \\
\Leftrightarrow & \text{for every } t \in D(G)\text{: } \mathcal{B}_{[x/t^{\mathcal{B}}]}(H) = 1 & \text{(conversion lemma)} \\
\Leftrightarrow & \text{for every } t \in D(G)\text{: } \mathcal{B}[x/t](H) = 1 & (\mathcal{B} \text{ is Herbrand structure}) \\
\Leftrightarrow & \mathcal{B}(\forall x\ H) = 1 & (U_{\mathcal{B}} = D(G)) \\
\Leftrightarrow & \mathcal{B} \models G &
\end{array}
$$

Löwenheim-Skolem's Theorem

Theorem: Every satisfiable formula of predicate logic has a model with a countable universe.

Proof: Let F be a formula, and let G be a sat-equivalent formula in Skolem form. Then for every set X:

F has a model with universe X

iff

G has a model with universe X.

$$
\begin{aligned}
F \text{ satisfiable} &\Rightarrow G \text{ satisfiable} \\
&\Rightarrow G \text{ has a Herbrand model } (X, I_1) \\
&\Rightarrow F \text{ has a model } (X, I_2) \\
&\Rightarrow F \text{ has a countable model}
\end{aligned}
$$

(Herbrand universes are countable)

Herbrand expansion

Let $F = \forall y_1 \forall y_2 \ldots \forall y_n F^*$ be a closed formula in Skolem form. The Herbrand expansion of F is the set of atomic formulas

$$E(F) = \{F^*[y_1/t_1][y_2/t_2] \ldots [y_n/t_n] \mid t_1, t_2, \ldots, t_n \in D(F)\}$$

Informally: the formulas of $E(F)$ are the result of substituting the variables of F^* by the terms of $D(F)$ in every possible way.

Gödel-Herbrand-Skolem's Theorem

Theorem: A closed formula F in Skolem form is satisfiable if and only if its Herbrand expansion $E(F)$ is satisfiable (in the sense of propositional logic).

Proof: It suffices to show: if $E(F)$ is satisfiable, then F has a Herbrand model.

Let F be of the form $\forall y_1 \forall y_2 \ldots \forall y_n F^*$. We have:

$\mathcal{A}$ is a Herbrand model of F

iff for every $t_1, t_2, \ldots, t_n \in D(F)$:

$$\mathcal{A}_{[y_1/t_1][y_2/t_2]\ldots[y_n/t_n]}(F^*) = 1$$

iff for every $t_1, t_2, \ldots, t_n \in D(F)$:

$$\mathcal{A}(F^*[y_1/t_1][y_2/t_2]\ldots[y_n/t_n]) = 1$$

iff for every $G \in E(F)$ we have $\mathcal{A}(G) = 1$

iff $\mathcal{A}$ is a model of $E(F)$

Herbrand's Theorem

Theorem: A closed formula F in Skolem form is unsatisfiable if and only if some finite subset of the Herbrand expansion of $E(F)$ is unsatisfiable.

Proof: Follows immediately from the Gödel-Herbrand-Skolem's Theorem and the Compactness Theorem.

Gilmore's Algorithm

Let F be closed formula in Skolem form and let $\{F_1, F_2, F_3, \ldots,\}$ be an enumeration of $E(F)$.

> Input: F
> $n := 0$;
> $\mathbf{repeat}$ $n := n + 1$;
> $\mathbf{until}$ $(F_1 \wedge F_2 \wedge \ldots \wedge F_n)$ is unsatisfiable;
> $\mathbf{report}$ "unsatisfiable" and $\mathbf{halt}$.

Semi-decidiability Theorems

Theorem:

(a) The unsatisfiability problem of predicate logic is semi-decidable.

(b) The validity problem of predicate logic is semi-decidable.

(c) The consequence problem of predicate logic is semi-decidable.

(d) The equivalence problem of predicate logic is semi-decidable.

Proof: (a) Gilmore's algorithm is a semi-decision algorithm.
(b) F valid iff $\neg F$ unsatisfiable.
(c) $F \models G$ iff $F \rightarrow G$ valid.
(d) $F \equiv G$ iff $F \leftrightarrow G$ valid.