The Compactness Theorem

Theorem: A set $\mathbf{S}$ of formulas is satisfiable iff every finite subset of $\mathbf{S}$ is satisfiable.

Equivalent formulation: A set $\mathbf{S}$ of formulas is unsatisfiable iff some finite subset of $\mathbf{S}$ is unsatisfiable.

Proof I

$\Rightarrow$: If $\mathbf{S}$ is satisfiable then every finite subset of $\mathbf{M}$ is satisfiable.

Trivial.

$\Leftarrow$: If every finite subset of $\mathbf{S}$ is satisfiable then $\mathbf{S}$ is satisfiable.

We prove that $\mathbf{S}$ has a model.

For every $n \geq 1$ let $\mathbf{S_n}$ be the subset of formulas of $\mathbf{S}$ containing only the atomic formulas $A_1, \ldots, A_n$.
(More precisely: not containing any occurrence of $A_{n+1}, A_{n+2}, \ldots$.)

Observe: We have $\mathbf{S_1} \subseteq \mathbf{S_2} \subseteq \mathbf{S_3} \ldots$

Proof II

Claim 1: Each of the sets $\mathbf{S_n}$ has a model $\mathcal{A}_n$.

Proof: Partition $\mathbf{S_n}$ into equivalence classes containing equivalent formulas. There are at most 2^{2^n} classes (why?). Pick a representative from each class. The set of all representatives is finite, and so by hypothesis it has a model $\mathcal{A}_n$, which is also a model of $\mathbf{S_n}$.

Claim 2: $\mathcal{A}_n$ is model not only of $\mathbf{S_n}$, but also of $\mathbf{S_1}, \ldots, \mathbf{S_{n-1}}$.
Proof: follows immediately from $\mathbf{S_1} \subseteq \mathbf{S_2} \subseteq \mathbf{S_3} \ldots$.

Proof III

Claim 3: Every assignment $\mathcal{A}$ satisfying the following property is a model of $\mathbf{S}$:

> For every $i \geq 1$ there is $j \geq i$ so that the restriction of $\mathcal{A}$ to $A_1, \ldots, A_i$ and the restriction of $\mathcal{A}_j$ to $A_1, \ldots, A_i$ coincide.

Proof: Since $j \geq i$ and $\mathcal{A}_j$ is model of $\mathbf{S_j}$, it is also model of $\mathbf{S_i}$. Since $\mathcal{A}$ and $\mathcal{A}_j$ coincide on $A_1, \ldots, A_i$, $\mathcal{A}$ is also model of $\mathbf{S_i}$. Thus, $\mathcal{A}$ is a model of each $\mathbf{S_i}$ and hence of $\mathbf{S}$.

Proof IV

Claim 4: There is a truth assignment $\mathcal{A}$ satisfying the condition of Claim 3.

Proof: We define $\mathcal{A}$ by means of an iterative procedure whose n-th iteration fixes $\mathcal{A}(A_n)$.

We maintain a set of indices I, initially $I := \mathbb{N}$.

At the n-th step, if there are infinitely many indices $i \in I$ such that $\mathcal{A}_i(A_n) = 1$, then

- set $\mathcal{A}(A_n) := 1$, and
- remove from I all indices i such that $\mathcal{A}_i(A_n) = 0$;

and otherwise

- set $\mathcal{A}(A_n) := 0$, and
- remove from I all indices i such that $\mathcal{A}_i(A_n) = 1$.