Technische Universität München
I7
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Logic

Exam July 11, 2014

Time: 120 minutes
If not stated otherwise, all answers have to be justified.

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1. Let the following propositional formula

$$F = (A \vee \neg B \vee \neg D \vee \neg E) \wedge (\neg B \vee C) \wedge B \wedge (\neg C \vee D) \wedge (\neg D \vee E)$$

be given.

a) Decide whether F is satisfiable by using the algorithm for Horn formulas discussed in the lecture.

b) How many models defined precisely on A, B, C, D, E does F have?

c) How many models defined precisely on A, B, C, D, E does $\neg F$ have?

Possible solution.

a) In the first round (before the loop) B is being marked. In round two C is being marked, in round three D is being marked, in round four E is being marked and in round five A is being marked. The marking algorithm outputs that F is satisfiable and computes the satisfying truth assignment $\mathcal{A}$, where $\mathcal{A}(A) = \mathcal{A}(B) = \mathcal{A}(C) = \mathcal{A}(D) = \mathcal{A}(E) = 1$.

b) We have shown in case the input formula is satisfiable that the marking algorithm computes a minimal model (with respect to set inclusion of the variables that are set to 1), hence $\mathcal{A}$ is the only model of F that is defined on A, B, C, D, E.

c) We have five variables, hence $2^5 = 32$ different truth assignments defined on A, B, C, D, E. Since F only has one such model, $\neg F$ has $32 - 1 = 31$ such models.

2. Let the following propositional formula

$$F = \neg\left(((A \rightarrow B) \wedge (B \rightarrow A)) \rightarrow (A \leftrightarrow B)\right)$$

be given.

a) Transform F into some equivalent formula G in conjunctive normal form by applying a sequence of equivalences introduced in the lecture (the name of these rules do not have to be specified).

b) Write G in clause form.

c) Give a derivation of $\Box$ from G (either as a sequence or as a tree).

d) Compute $\mathsf{Res}^0(G)$ and $\mathsf{Res}^1(G)$ and determine the set $\{i \in \mathbb{N} \mid \Box \in \mathsf{Res}^i(G)\}$.

e) How many models defined precisely on A, B, C does $F \vee (A \to B)$ have?

Possible solution.

a) Here, we really went step by step. In case a solution took more than one steps at once we will not be too strict about this.

$$\begin{aligned}
F &= \neg (((A \to B) \wedge (B \to A)) \to (A \leftrightarrow B)) \\
&\equiv \neg (((\neg A \vee B) \wedge (B \to A)) \to (A \leftrightarrow B)) \\
&\equiv \neg (((\neg A \vee B) \wedge (\neg B \vee A)) \to (A \leftrightarrow B)) \\
&\equiv \neg (((\neg A \vee B) \wedge (\neg B \vee A)) \to ((A \wedge B) \vee (\neg A \wedge \neg B))) \\
&\equiv \neg (\neg ((\neg A \vee B) \wedge (\neg B \vee A)) \vee ((A \wedge B) \vee (\neg A \wedge \neg B))) \\
&\equiv \neg\neg ((\neg A \vee B) \wedge (\neg B \vee A)) \wedge \neg((A \wedge B) \vee (\neg A \wedge \neg B)) \\
&\equiv (\neg A \vee B) \wedge (\neg B \vee A) \wedge \neg((A \wedge B) \vee (\neg A \wedge \neg B)) \\
&\equiv (\neg A \vee B) \wedge (\neg B \vee A) \wedge \neg(A \wedge B) \wedge \neg(\neg A \wedge \neg B) \\
&\equiv (\neg A \vee B) \wedge (\neg B \vee A) \wedge (\neg A \vee \neg B) \wedge \neg(\neg A \wedge \neg B) \\
&\equiv (\neg A \vee B) \wedge (\neg B \vee A) \wedge (\neg A \vee \neg B) \wedge (\neg\neg A \vee \neg\neg B) \\
&\equiv (\neg A \vee B) \wedge (\neg B \vee A) \wedge (\neg A \vee \neg B) \wedge (\neg\neg A \vee B) \\
&\equiv (\neg A \vee B) \wedge (\neg B \vee A) \wedge (\neg A \vee \neg B) \wedge (A \vee B) \\
&= G
\end{aligned}$$

b) G in clause form:

$$G = \Big\{\{\neg A, B\}, \{\neg B, A\}, \{\neg A, \neg B\}, \{A, B\}\Big\}$$

c) Derivation of $\Box$ as a sequence:

(1) $\{\neg A, B\}$ is a clause of G
(2) $\{\neg A, \neg B\}$ is a clause of G
(3) $\{\neg A\}$ is a resolvent of (1) and (2)
(4) $\{\neg B, A\}$ is a clause of G
(5) $\{A, B\}$ is a clause of G
(6) $\{A\}$ is a resolvent of (4) and (5)
(7) $\Box$ is a resolvent of (3) and (6)

d)
- $\mathsf{Res}^0(G) = G$
- $\mathsf{Res}^1(G) = \mathsf{Res}^0(G) \cup \Big\{\{A, \neg A\}, \{B, \neg B\}, \{\neg A\}, \{B\}, \{\neg B\}, \{A\}\Big\}$

- Since $\{A\}, \{\neg A\} \in \mathsf{Res}^1(G)$ we surely have $\square \in \mathsf{Res}^2(G)$ and hence $\square \in \mathsf{Res}^i(G)$ for each $i \geq 2$ by monotonicity of Res. Hence

$$\{i \in \mathbb{N} \mid \square \in \mathsf{Res}^i(G)\} = \{i \in \mathbb{N} \mid i \geq 2\} = \{2, 3, 4, \ldots\}.$$

e) Since $\square \in \mathsf{Res}^*(G)$ it follows F is unsatisfiable, thus $F \equiv 0$ and we get $F \vee (A \rightarrow B) \equiv A \rightarrow B$. Over the variables A, B formula $A \vee B$ has three models, over A, B, C it has 6 models.

3. Let the following formula

$$F = \forall x \forall y \exists z \left(\left(\neg(x = y) \rightarrow \neg R(f(x), f(y)) \right) \wedge \exists x (R(b, z) \wedge R(x, y)) \right)$$

of predicate logic be given.

a) Give a model $\mathcal{A}$ of F such that $|U_{\mathcal{A}}|$ is minimal (without justification).

b) Skolemize the formula F into some formula G. In every step, state how the formula was transformed and whether semantic equivalence or only equi-satisfiability holds (the name of the rules do not have to be specified).

c) How many elements does the Herbrand universe of G have?

d) Give five elements of the Herbrand universe of G.

Possible solution.

a) The suitable structure $\mathcal{A}$ with $U_{\mathcal{A}} = \{1\}$, $R^{\mathcal{A}} = \{(1, 1)\}$, $b^{\mathcal{A}} = 1$ and $f^{\mathcal{A}}(1) = 1$ is obviously a model of F.

b)

$$\begin{aligned}
F &= \forall x \forall y \exists z \left(\left(\neg(x = y) \rightarrow \neg R(f(x), f(y)) \right) \wedge \exists x (R(b, z) \wedge R(x, y)) \right) \\
&\equiv \forall x \forall y \exists z \left(\left(\neg(x = y) \rightarrow \neg R(f(x), f(y)) \right) \wedge \exists w (R(b, z) \wedge R(w, y)) \right) \\
&\equiv \forall x \forall y \exists z \exists w \left(\left(\neg(x = y) \rightarrow \neg R(f(x), f(y)) \right) \wedge R(b, z) \wedge R(w, y) \right) \\
&\equiv_s \forall x \forall y \exists w \left(\left(\neg(x = y) \rightarrow \neg R(f(x), f(y)) \right) \wedge R(b, g(x, y)) \wedge R(w, y) \right) \\
&\equiv_s \forall x \forall y \left(\left(\neg(x = y) \rightarrow \neg R(f(x), f(y)) \right) \wedge R(b, g(x, y)) \wedge R(h(x, y), y) \right)
\end{aligned}$$

c) The Herbrand universe is infinite since we have at least one functional symbol of arity at least one.

d) The Herbrand universe contains all terms over b, f, g and h, among them for instance b,$f(b)$,$g(b, b)$,$f(f(g(b, b)))$, $g(f(b), b)$.

4. Let the signature $S = \{R, f\}$ be given, where R is a binary relational symbol (sometimes it helps reading R as an edge relation of a directed graph) and where f is a unary functional symbol.

a) Give a formula F over S with equality such that for each suitable structure $\mathcal{A}$ it holds that $\mathcal{A} \models F$ if and only if R is an equivalence relation with precisely two equivalence classes (no justification necessary).

b) Prove that $\exists x \forall y R(x, y) \models \forall x \exists y R(y, x)$.

c) Give a satisfiable formula F over S without equality that has only infinite models (only the formula is required).

d) Give a satisfiable formula F over S with equality that has precisely four models (that are defined on S and up to isomorphism). Draw the four models (no further explanation required).

Possible solution.

a)

$$F = \forall x \forall y \forall z \Big((R(x,x) \wedge (R(x,y) \to R(y,x)) \wedge (R(x,y) \wedge R(y,z) \to R(x,z)) \Big) \wedge$$

$$\forall x \forall y \forall z \Big(R(x,y) \vee R(x,z) \vee R(y,z) \Big) \wedge \exists x \exists y \neg R(x,y)$$

b) We have to show that suitable model $\mathcal{A}$ of $\exists x \forall y R(x,y)$ is also a model of $\forall x \exists y R(y,x)$. Assume $\mathcal{A} \models \exists x \forall R(x,y)$. Hence there exists some $a \in U_{\mathcal{A}}$ such that $(a,b) \in E^{\mathcal{A}}$ for each $b \in U_{\mathcal{A}}$. Let $c \in U_{\mathcal{A}}$ be arbitary, then we know that in particular $(a,c) \in R^{\mathcal{A}}$. Hence for each $c \in U_{\mathcal{A}}$ we have $\mathcal{A}_{[x/c]} \models \exists y R(y,x)$. Thus, $\mathcal{A} \models \forall x \exists y R(y,x)$.

c)

$$F = \forall x \forall y \forall z \Big((R(x,y) \wedge R(y,z) \to R(x,z)) \wedge \neg R(x,x) \wedge R(x, f(x)) \Big)$$

(expressing R is an irreflexive and transitive binary relation that contains f interpreted as a binary relation).

d)

$$F = \forall x \forall y \Big(\neg R(x,y) \wedge x = f(x) \Big) \wedge \forall x_1 \forall x_2 \forall x_3 \forall x_4 \forall x_5 \bigvee_{i,j \in \{1,2,3,4,5\}, i \neq j} x_i = x_j$$

expressing that R is always empty, f always has self-loops at each element and that there are at most 4 elements in the universe. Hence the structures $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4$ are precisely the models of F, where for each $i \geq 1$ we set $U_{\mathcal{A}_i} = \{1, \ldots, i\}$, $R^{\mathcal{A}_i} = \emptyset$ and $f^{\mathcal{A}_i}(j) = j$ for each $j \in U_{\mathcal{A}_i}$.

5. Confirm or refute the following statements. Always provide a short justification of your answer.

a) There is a formula F of predicate logic that has a model $\mathcal{A}$ with $U_{\mathcal{A}} = \mathbb{R}$ (the real numbers).

b) For any two isomorphic structures $\mathcal{A}$ and $\mathcal{B}$ it holds that for each $k \geq 17$ Duplicator has a winning strategy in $\mathcal{G}_k(\mathcal{A}, \mathcal{B})$.

c) For each structure $\mathcal{A}$ we have that if $U_{\mathcal{A}}$ is infinite, then $Th(\mathcal{A})$ is not decidable.

d) For each propositional formulas F, G and H we have that $F \wedge G \models H$ if and only if $(F \rightarrow G) \rightarrow H$ is valid.

e) $\{A, B, (A \rightarrow \neg B)\} \models (A \rightarrow \neg B) \rightarrow B$.

f) For any two structures $\mathcal{A}$ and $\mathcal{B}$ over the same signature it holds that if $U_{\mathcal{A}} \subseteq U_{\mathcal{B}}$, then $Th(\mathcal{A}) \subseteq Th(\mathcal{B})$.

g) For each formula F with $\mathsf{qr}(F) = k$ there exists a formula G with $\mathsf{qr}(G) = k+1$ and $F \equiv G$.

h) The satisfiability problem for propositional formulas in disjunctive normal form can be solved in polynomial time.

Possible solution.

a) Yes, take any valid formula, for instance the formula $\forall x(x = x)$.

b) Yes, if $\mathcal{A} \simeq \mathcal{B}$, then $\mathcal{A}$ cannot be distinghuished by any formula in predicate logic. By the Ehrenfeucht-Fraïssé theorem in particular for each $k \geq 0$ Duplicator has a winning strategy for $\mathcal{G}_k(\mathcal{A}, \mathcal{B})$, in particular for $k \geq 17$.

c) No, we have proven by quantifier elimination that linear arithmetic, i.e. $Th(\mathbb{Q}, 0, 1, +, c \cdot (c \in \mathbb{Q}), <)$ is decidable.

d) No, take $F = H = 0$ and $G = 1$, then surely $F \wedge G \models H$ but $(F \rightarrow G) \rightarrow H \equiv 0$ is surely not valid.

e) Yes, obviously $\{A, B, (A \rightarrow \neg B)\}$ is unsatisfiable, hence $\{A, B, (A \rightarrow \neg B)\} \models F$ for each formula F.

f) No, if $\mathcal{A}$'s universe is a singleton then $F = \exists x \forall y(x = y) \in Th(\mathcal{A})$ but if $\mathcal{B}$'s universe has at least two elements, then $F \notin Th(\mathcal{B})$.

g) Yes, choose $G = F \wedge \forall x_1 \cdots \forall x_k (x_1 = x_1)$.

h) Yes, because a formula $F = \bigvee_{i=1}^{\ell} F_i$ in disjunctive normal form is satisfiable if and only if at least one F_i is satisfiable and one such F_i is satisfiable if and only if it does not contain an atomic formula both negatively and positively. The latter is easily verifiable in polynomial time.

6. a) Let S be an arbitrary finite signature with relational symbols only. Prove that the property

$$P = \{\mathcal{A} : |U_{\mathcal{A}}| \in \mathbb{N} \text{ is a prime number}\}$$

is not expressible in predicate logic over S by applying the methodology theorem. When showing the existence of a winning Duplicator strategy, only the winning strategy is required (and not any proof why it is winning).

b) Let us fix the signature $S = \{R\}$, where R is a binary relational symbol. Let the property

$$P = \{\mathcal{A} \mid \mathcal{A} \text{ has a directed } R\text{-cycle of length } 3\}$$

be given.

(i) Give some formula F of predicate logic that expresses P with $\mathsf{qr}(F) = 3$ (only the formula is required).

(ii) Give two suitable structures $\mathcal{A}$ and $\mathcal{B}$ (drawing them suffices and no further justification necessary) such that

- $\mathcal{A}$ satisfies P and $\mathcal{B}$ does not satisfy P.
- Duplicator has a winning strategy in $\mathcal{G}_2(\mathcal{A}, \mathcal{B})$.

Possible solution.

a) By the methodology theorem it suffices to construct for each $k \geq 0$ two structures $\mathcal{A}_k$ and $\mathcal{B}_k$ such that

(i) $\mathcal{A}_k$ satisfies P and $\mathcal{B}_k$ does not satisfy $\mathcal{B}_k$.

(ii) Duplicator has a winning strategy in $\mathcal{G}_k(\mathcal{A}_k, \mathcal{B}_k)$.

We choose $\mathcal{A}_k$ as $U_{\mathcal{A}_k} = \{1, \ldots, m\}$, where m is the smallest prime greater or equal to k and $R^{\mathcal{A}_k} = \emptyset$ for each $R \in S$. We choose $\mathcal{B}_k$ as $U_{\mathcal{A}_k} = \{1, \ldots, 4k\}$ and also $R^{\mathcal{B}_k} = \emptyset$ for each $R \in S$. After having played i rounds with the pebbles $\{(a_1, b_1), \ldots, (a_i, b_i)\}$ Duplicator's winning strategy for the $(i+1)^{\text{st}}$ round is as follows:

- If some element a with $a = a_h$ for some $h \in \{1, \ldots i\}$ is played, answer b_h.
- If some element b with $b = b_h$ for some $h \in \{1, \ldots i\}$ is played, answer a_h.
- If some element $a \in U_{\mathcal{A}} \setminus \{a_1, \ldots, a_i\}$ is played, answer with some $b \in U_{\mathcal{B}_k} \setminus \{b_1, \ldots, b_i\}$.
- If some element $b \in U_{\mathcal{B}} \setminus \{b_1, \ldots, b_i\}$ is played, answer with some $a \in U_{\mathcal{A}_k} \setminus \{a_1, \ldots, a_i\}$.

b) (i) $F = \exists x \exists y \exists z (R(x, y) \wedge R(y, z) \wedge R(z, x))$ expresses P.

(ii) Choose $\mathcal{A}$ as $U_{\mathcal{A}} = \{1, 2, 3\}$ with $R^{\mathcal{A}} = \{(1, 2), (2, 1), (2, 3), (3, 2), (1, 3), (3, 1)\}$ and $\mathcal{B}$ as $U_{\mathcal{B}} = \{a, b\}$ with $R^{\mathcal{B}} = \{(a, b), (b, a)\}$.