Technische Universität München
I7
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Logic

Exam July 11, 2014

Time: 120 minutes
Total number of points: **60**
If not stated otherwise, all answers have to be justified.

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1. (8=4+2+2)

Let the following propositional formula

$$F = (A \lor \neg B \lor \neg D \lor \neg E) \land (\neg B \lor C) \land B \land (\neg C \lor D) \land (\neg D \lor E)$$

be given.

a) Decide whether F is satisfiable by using the algorithm for Horn formulas discussed in the lecture.

b) How many models defined precisely on A, B, C, D, E does F have?

c) How many models defined precisely on A, B, C, D, E does $\neg F$ have?

2. (11=5+1+2+2+1)

Let the following propositional formula

$$F = \neg(((A \to B) \land (B \to A)) \to (A \leftrightarrow B))$$

be given.

a) Transform F into some equivalent formula G in conjunctive normal form by applying a sequence of equivalences introduced in the lecture (the name of these rules do not have to be specified).

b) Write G in clause form.

c) Give a derivation of $\Box$ from G (either as a sequence or as a tree).

d) Compute $\mathsf{Res}^0(G)$ and $\mathsf{Res}^1(G)$ and determine the set $\{i \in \mathbb{N} \mid \Box \in \mathsf{Res}^i(G)\}$.

e) How many models defined precisely on A, B, C does $F \lor (A \to B)$ have?

3. (7=2+3+1+1)

Let the following formula

$$F = \forall x \forall y \exists z \left(\left(\neg(x = y) \to \neg R(f(x), f(y)) \right) \land \exists x (R(b, z) \land R(x, y)) \right)$$

of predicate logic be given.

a) Give a model $\mathcal{A}$ of F such that $|U_{\mathcal{A}}|$ is minimal (without justification).

b) Skolemize the formula F into some formula G. In every step, state how the formula was transformed and whether semantic equivalence or only equi-satisfiability holds (the name of the rules do not have to be specified).

c) How many elements does the Herbrand universe of G have?

d) Give five elements of the Herbrand universe of G.

4. (9=2+2+2+3)

Let the signature $S = \{R, f\}$ be given, where R is a binary relational symbol (sometimes it helps reading R as an edge relation of a directed graph) and where f is a unary functional symbol.

a) Give a formula F over S with equality such that for each suitable structure $\mathcal{A}$ it holds that $\mathcal{A} \models F$ if and only if R is an equivalence relation with precisely two equivalence classes (no justification necessary).

b) Prove that $\exists x \forall y R(x, y) \models \forall x \exists y R(y, x)$.

c) Give a satisfiable formula F over S without equality that has only infinite models (only the formula is required).

d) Give a satisfiable formula F over S with equality that has precisely four models (that are defined on S and up to isomorphism). Draw the four models (no further explanation required).

5. (16=8×2)

Confirm or refute the following statements. Always provide a short justification of your answer.

a) There is a formula F of predicate logic that has a model $\mathcal{A}$ with $U_{\mathcal{A}} = \mathbb{R}$ (the real numbers).

b) For any two isomorphic structures $\mathcal{A}$ and $\mathcal{B}$ it holds that for each $k \geq 17$ Duplicator has a winning strategy in $\mathcal{G}_k(\mathcal{A}, \mathcal{B})$.

c) For each structure $\mathcal{A}$ we have that if $U_{\mathcal{A}}$ is infinite, then $Th(\mathcal{A})$ is not decidable.

d) For each propositional formulas F, G and H we have that $F \wedge G \models H$ if and only if $(F \to G) \to H$ is valid.

e) $\{A, B, (A \to \neg B)\} \models (A \to \neg B) \to B$.

f) For any two structures $\mathcal{A}$ and $\mathcal{B}$ over the same signature it holds that if $U_{\mathcal{A}} \subseteq U_{\mathcal{B}}$, then $Th(\mathcal{A}) \subseteq Th(\mathcal{B})$.

g) For each formula F with $\mathsf{qr}(F) = k$ there exists a formula G with $\mathsf{qr}(G) = k + 1$ and $F \equiv G$.

h) The satisfiability problem for propositional formulas in disjunctive normal form can be solved in polynomial time.

6. a) **(9=5+(1+3))**

Let S be an arbitrary finite signature with relational symbols only. Prove that the property

$$\{\mathcal{A} : |U_{\mathcal{A}}| \in \mathbb{N} \text{ is a prime number}\}$$

is not expressible in predicate logic over S by applying the methodology theorem. When showing the existence of a winning Duplicator strategy, only the winning strategy is required (and not any proof why it is winning).

b) Let us fix the signature $S = \{R\}$, where R is a binary relational symbol. Let the property

$$P = \{\mathcal{A} \mid \mathcal{A} \text{ has a directed } R\text{-cycle of length } 3\}$$

be given.

(i) Give some formula F of predicate logic that expresses P with $\mathsf{qr}(F) = 3$ (only the formula is required).

(ii) Give two suitable structures $\mathcal{A}$ and $\mathcal{B}$ (drawing them suffices and no further justification necessary) such that

- $\mathcal{A}$ satisfies P and $\mathcal{B}$ does not satisfy P.
- Duplicator has a winning strategy in $\mathcal{G}_2(\mathcal{A}, \mathcal{B})$.