Technische Universität München
I7
Stefan Göller ST 2014

Logic

Exercise Sheet 7

Discussion: June 12, 2014

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1. Let the following two axioms be given:

(1) $F \to (G \to F)$
(4) $F \to (\neg F \to G)$

Which of the following deductions in the Hilbert calculus are correct if the two axioms from above are the only ones allowed:

a) i.) $\neg A \vdash A \to (\neg A \to B)$
   ii.) $\neg A \vdash \neg A$
   iii.) $\neg A \vdash \neg(\neg A \to B)$

b) i.) $\neg A, B \vdash B$
   ii.) $\neg A, B \vdash \neg A$
   iii.) $\neg A, B \vdash \neg A \to B$

c) i.) $A \vdash \neg\neg A \to (A \to \neg\neg A)$
   ii.) $A \vdash \neg\neg A$
   iii.) $A \vdash A \to \neg\neg A$

d) i.) $A \vdash A \to (B \to A)$
   ii.) $A \vdash A$
   iii.) $A \vdash B \to A$

2. Using the complete set of axioms as presented in the lecture, give a deduction of $\neg\neg A \vdash A$ and of $A \vdash B \to (A \to A)$.

3. Provide a quantifier elimination procedure for $Th(\mathbb{N}, 0, 1, +, <, =)$ also known as Presburger arithmetic.

4. Let S be a signature. The *spectrum* of a formula F that is suitable for S is the set of all $n \in \mathbb{N}$ such that F has a model with universe of size n.

Show the following for predicate logic with equality:

a) $\emptyset$ and $\mathbb{N} \setminus \{0\}$ are the spectrum of some formula over the empty signature.

b) Every finite set is the spectrum of some sentence over the empty signature.

c) Every set $U \subseteq \mathbb{N} \setminus \{0\}$ such that $\mathbb{N} \setminus U$ is finite is the spectrum of a sentence over the empty signature.

d) There exists a sentence F over the signature $\{R\}$, where R is a binary relational symbol such that the spectrum of F is set of all positive even naturals.