Technische Universität München
I7
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Logic

Exercise Sheet 3

Discussion: May 8, 2014

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1. Let the boolean functions $F = (x \vee y) \rightarrow z$ and $G = x \leftrightarrow (y \wedge z)$ be given. Let us furthermore fix the variable order $x < y < z$.

- Draw the BDDs for F and G.
- During the lecture an algorithm implementing the OR-operation on BDDs has been presented. Use this algorithm to construct a BDD for $F \vee G$.
- Modify this algorithm to receive the BDD for $F \wedge G$.
- Let $H = ((z \leftrightarrow (x \oplus y)) \wedge (w \leftrightarrow (y \vee z))) \vee x \vee (u \leftrightarrow (x \vee w))$, where the operator $\oplus$ denotes exclusive OR: $F \oplus G = \neg(F \leftrightarrow G)$. Draw some BDD for H. How many models with domain $\{u, w, x, y, z\}$ exist for H?
  Hint: For each node give the number of satisfying assignments for the subtree starting at that node. Start with the "lowest" one.

2. Kőnig's Lemma states the following: A directed tree $T = (V, <)$ with root r, where each node has only finitely many children, is finite if and only if there is no infinite path from r.

Prove Kőnig's Lemma for countable graphs by applying the compactness theorem for propositional logic. Some hints:

- Introduce a variable x_v for each vertex $v \in V$.
- Let $V_i = \{v \in V \mid r <^i v\}$ denote the set of all nodes having distance i from r for each $i \geq 0$.
- Choose a set of formulas Γ such that each finite subset of Γ is satisfiable and that Γ is satisfiable if and only if T contains an infinite path from r.

3. Is the following formula valid, is it satisfiable?

a) $(\forall x \exists y \exists z R(x, y) \wedge R(x, z) \wedge \neg(y = z)) \rightarrow \exists x \exists y \exists z (R(x, z) \wedge R(y, z) \wedge \neg(y = z))$.

4. Which of the following structures is a model of the formula F?

$$F = \exists x \exists y \exists z (P(x, y) \wedge P(z, y) \wedge P(x, z) \wedge \neg P(z, x)).$$

a) $U_{\mathcal{A}} = \mathbb{N}$ and $P^{\mathcal{A}} = \{(m, n) \mid m, n \in \mathbb{N}, m < n\}$.

b) $U_{\mathcal{A}} = \mathbb{N}$ and $P^{\mathcal{A}} = \{(m, m+1) \mid m \in \mathbb{N}\}$.

c) $U_{\mathcal{A}} = 2^{\mathbb{N}}$ and $P^{\mathcal{A}} = \{(A, B) \mid A, B \subseteq \mathbb{N}, A \subseteq B\}$.

5. Prove the following. Let $F = QxG$ be a formula where $Q \in \{\forall, \exists\}$. Let y be a variable that does not occur free in G. Then $F \equiv QyG[x/y]$.