Predicate logic with equality

Predicate logic
$+$
distinguished predicate symbol "=" of arity 2.

Semantics : a structure $\mathcal{A}$ of predicate logic with equality always maps the predicate symbol = to the identity relation:

$$\mathcal{A}(=) \;=\; \{(d,d) \mid d \in U_{\mathcal{A}}\}\,.$$

Expressivity

Fact: A structure is model of $\exists x \forall y\;\; x{=}y$ iff its universe is a singleton.

Theorem: Every satisfiable formula of predicate logic has an infinite countable model.

Proof: Let F satisfiable. We assume w.l.o.g. that $F = \forall x_1 \ldots \forall x_n F^*$ and the variables occurring in F^* are exactly $x_1, \ldots, x_n$. (If necessary bring F into Skolem form). We consider two cases:
$n = 0$. Exercise.
$n > 0$. Let $G = \forall x_1 \ldots \forall x_n F^*[x_1/f(x_1)]$, where f is a function symbol that does not occur in F^*. G ist satisfiable (why?) and $D(G)$ is infinite. It follows from the fundamental theorem that G has an infinite model.

Modelling equality

Let F be a formula of predicate logic with equality, and let Eq be a predicate symbol that does not occur in F.
Let G_F be the conjunction of the following formulas:

$\forall x\; Eq(x,x)$

$\forall x \forall y\; (Eq(x,y) \rightarrow Eq(y,x))$

$\forall x \forall y \forall z\; ((Eq(x,y) \wedge Eq(y,z)) \rightarrow Eq(x,z))$

$\forall x_1 \ldots \forall x_n \forall y\; (Eq(x_i,y) \rightarrow Eq(f(x_1,\ldots,x_i,\ldots x_n), f(x_1,\ldots,y,\ldots,x_n)))$

for every function symbol f of F and every $1 \leq i \leq n$

$\forall x_1 \ldots \forall x_n \forall y (Eq(x_i,y) \rightarrow (P(x_1,\ldots,x_i,\ldots,x_n) \leftrightarrow P(x_1,\ldots,y,\ldots,x_n)))$

for every predicate symbol P of F und and every $1 \leq i \leq n$

Let H_F be the formula obtained from F by substituting every occurrence of "=" by "Eq".

Theorem: The formulas F and $G_F \wedge H_F$ are sat-equivalent.

Proof: We show that if $G_F \wedge H_F$ is satisfiable then F is satisfiable. Let $\mathcal{A}$ be a model of $G_F \wedge H_F$. Then $Eq^{\mathcal{A}}$ is an equivalence relation. For every $d \in U_{\mathcal{A}}$ let $[d]$ be the equivalence class of d. Define the structure $\mathcal{B}$ as follows:

- $U_{\mathcal{B}} = \{[d] \mid d \in U_{\mathcal{A}}\}$.
- For every function symbol f of F:
  $f^{\mathcal{B}}([d_1],\ldots,[d_n]) = [f^{\mathcal{A}}(d_1,\ldots,d_n)]$
- For every predicate symbol P of F:
  $([d_1],\ldots,[d_n]) \in P^{\mathcal{B}}$ iff $(d_1,\ldots,d_n) \in P^{\mathcal{A}}$

$\mathcal{B}$ is well defined because $\mathcal{A} \models G_F$.

Since $\mathcal{A} \models H_F$ we get $\mathcal{B} \models F$.

An application

Theorem: Every formula without function symbols of the form $\forall x \exists u \forall y\ F^*$ is sat-equivalent to a formula without function symbols of the form $\forall x \forall y \forall z \exists v\ G^*$.

Proof: Let P be a predicate symbol not occurring in F. Let

$$H = \forall x \forall y \forall z \exists v\ (P(x,v) \wedge (P(x,y) \rightarrow y\text{=}v) \wedge (P(x,z) \rightarrow F^*[u/z]))$$

It is easy to see that the original formula and H are sat-equivalent

Replace $y = v$ by $Eq(y,v)$ in H, add the formulas expressing that Eq is a congruence, convert the resulting formula into prenex form, and let G be the result.

We have that H and G are sat-equivalent, and so the original formula and G are sat-equivalent.