Source

G.S. Boolos, J.P. Burgess, R.C Jeffrey:
Computability and Logic. Cambridge University Press 2002.

Theories

A signature is a (finite or infinite) set of predicate and function symbols. We fix a signature S. "Formula" means now "formula over the signature S".

A theory is a set of formulas T closed under consequence, i.e., if $F_1, \ldots, F_n \in T$ and $\{F_1, \ldots, F_n\} \models G$ then $G \in T$.

Fact: Let $\mathcal{A}$ be a structure suitable for S. The set F of formulas such that $\mathcal{A}(F) = 1$ is a theory.

We call them model-based theories.

Fact: Let $\mathcal{F}$ be a set of closed formulas. The set F of formulas such that $\mathcal{F} \models F$ is a theory.

The signature of arithmetic

The signature S_A of arithmetic contains:

- a constant 0,
- a unary function symbol s,
- two binary function symbols $+$ and $\cdot$, and
- a binary predicate symbol $<$.

(slight change over previous definitions)

`Arith` is the theory containing the set of closed formulas over S_A that are true in the canonical structure.

`Arith` contains "all the theorems of calculus".

Decidability, consistency, completeness, . . .

A set $\mathcal{F}$ of formulas is decidable if there is an algorithm that decides for every formula F whether $F \in \mathcal{F}$ holds.

Let T be a theory.

T is decidable if it is decidable as a set of formulas.

T is consistent if for every closed formula F either $F \notin T$ or $\neg F \notin T$.

T is complete if for every closed formula F either $F \in T$ or $\neg F \in T$.

T is (finitely) axiomatizable if there is a (finite) decidable set $\mathcal{X} \subseteq T$ of axioms such that every closed formula of T is a consequence of $\mathcal{X}$.

Conventions and notation

In the following: set of axioms = decidable set of formulas over S_A

$T_{\mathcal{X}}$ denotes the theory of all closed formulas that are consequences of a set $\mathcal{X}$ of axioms.

Basic facts

Fact: Every theory contains all valid formulas (because they are consequences of the empty set).

Fact: Model-based theories (like `Arith`) are consistent and complete.

Fact: T is consistent iff there is a formula F such that $F \notin T$.
Proof: If T is consistent then $F \notin T$ for some F by definition.
If T is inconsistent, then there exists a formula F such that $F \in T$ and $\neg F \in T$. Let G be an arbitrary closed formula. Since $F, \neg F \models G$ and T is closed under consequence, we have $G \in T$.

Basic facts

Lemma: If T is axiomatizable and complete, then T is decidable.

Proof: If T inconsistent then T contains all closed formulas, and the algorithm that answers "$F \in T$" for every input F decides T.
If T consistent, let consider the following algorithm:

- Input: F
  Enumerate all syntactic consequences of the axioms of T, and for each new syntactic consequence G do:
  - If $G = F$ halt with "$F \in T$"
  - If $G = \neg F$ halt with "$F \notin T$"

Observe: the syntactic consequences of the axioms can be enumerated.
We prove this algorithm is correct:

- If algorithm answers "$F \in T$", then $F \in T$.
  If algorithm answers "$F \in T$", then F is syntactic consequence, and so consequence of the axioms. Since T is a theory, $F \in T$.
- If algorithm answers "$F \notin T$", then $F \notin T$.
  If algorithm answers "$F \in T$", then $\neg F$ is consequence of the axioms and so $\neg F \in T$. By consistency, $F \notin T$.
- The algorithm terminates.
  Since T is complete, either $F \in T$ or $\neg F \in T$.
  Assume w.l.og. $F \in T$.
  Since T is axiomatizable, F is a consequence of the axioms.
  So F is a syntactic consequence of the axioms.
  So eventually $G := F$ and the algorithm terminates.

Basic facts

Theorem: Arith is undecidable.

Proof: By reduction from the halting problem, similar to undecidability proof for validity of predicate logic.

Theorem: Arith is not axiomatizable.

Proof: Since Arith is undecidable, consistent, and complete, it is not axiomatizable (see Lemma).

Gödel's first incompleteness theorem

Theorem: Let $\mathcal{X}$ be any set of axioms such that $\mathcal{X} \subseteq$ Arith. Then the theory $T_{\mathcal{X}}$ is incomplete.

Proof: Since Arith is not axiomatizable, there is a formula $F \in$ Arith such that $\mathcal{X} \not\models F$ and so $F \notin T_{\mathcal{X}}$.

Assume now $\neg F \in T_{\mathcal{X}}$. Then $\mathcal{X} \models \neg F$ and since $\mathcal{X} \subseteq$ Arith we get $\neg F \in$ Arith, contradicting $F \in$ Arith.

So $F \notin T_{\mathcal{X}}$ and $\neg F \notin T_{\mathcal{X}}$, which proves that $T_{\mathcal{X}}$ is incomplete.

Gödel's first incompleteness theorem

Observe: $F \in$ Arith, i.e., F is true in the canonical structure, but its truth cannot be proved using any set $\mathcal{X}$ of axioms (unless some axiom is itself not true!)

In other words: for every set of true axioms, there are true formulas that cannot be deduced from the axioms

But we have no idea how such formulas look like ...

Goal: given a set of axioms $\mathcal{X} \subseteq$ Arith, construct a formula $F \in$ Arith such that $F \notin T_{\mathcal{X}}$

Minimal arithmetic

Minimal arithmetic Q is the axiom-based theory over S_A having the following axioms:

(Q1)	$\forall x$	$\neg(0 = s(x))$
(Q2)	$\forall x \forall y$	$s(x) = s(y) \rightarrow x = y$
(Q3)	$\forall x$	$x + 0 = x$
(Q4)	$\forall x \forall y$	$x + s(y) = s(x + y)$
(Q5)	$\forall x$	$x \cdot 0 = 0$
(Q6)	$\forall x \forall y$	$x \cdot s(y) = (x \cdot y) + x$
(Q7)	$\forall x$	$\neg(x < 0)$
(Q8)	$\forall x \forall y$	$x < s(y) \leftrightarrow (x < y \vee x = y)$
(Q9)	$\forall x \forall y$	$x < y \vee x = y \vee y < x$

Peano arithmetic

Peano arithmetic P is the axiom-based theory over S_A having Q1-Q9 as axioms plus all closed formulas of the form

$$\text{(I)}\quad \forall \mathbf{y} \quad F(0,\mathbf{y}) \wedge \forall x\,(F(x,\mathbf{y}) \rightarrow F(s(x),\mathbf{y})) \rightarrow \forall x\, F(x,\mathbf{y})$$

where $\mathbf{y} = (y_1, \ldots y_n)$.

Observe: I is an axiom scheme; the set of axioms of P is infinite but decidable.

Some theorems of Q (and P)

$\neg(0 = s^n(0))$ for every $n \geq 1$

$\neg(s^n(0) = s^m(0))$ for every $n, m \geq 1$, $n \neq m$

$\forall x \quad x < 1 \leftrightarrow x = 0$

$\forall x \quad x < s^{n+1}(0) \leftrightarrow (x = 0 \vee x = s(0) \vee \ldots \vee x = s^n(0))$

$s^n(0) + s^m(0) = s^l(0)$ for every $n, m, l \geq 1$ such that $n + m = l$

$s^n(0) \cdot s^m(0) = s^l(0)$ for every $n, m, l \geq 1$ such that $n \cdot m = l$

Gödel encodings

A Gödel encoding is an injective function that maps every formula over S_A to a natural number called its Gödel number.

Simple Gödel encoding: assign to each symbol of the formula its ASCII code, assign to a formula the concatenation of the ASCII codes of its symbols.

Gödel encodings

Example (Wikipedia): the formula

$$x = y \rightarrow y = x$$

written in ASCII as

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x=y => y=x
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corresponds to the sequence

120-061-121-032-061-062-032-121-061-120

of ASCII codes, and so it is assigned the number

120061121032061062032121061120

Gödel's Gödel encoding

Let p_n denote the n-th prime number.

Gödel's encoding assigns to each symbol λ a number $g(\lambda)$, and to a sequence $\lambda_1 \cdots \lambda_n$ of symbols the number

$$2^{g(\lambda_1)} \cdot 3^{g(\lambda_2)} \cdot 5^{g(\lambda_3)} \cdot \ldots \cdot p_n^{g(\lambda_n)}$$

What are Gödel encodings good for?

A formula $F(x)$ over S_A with a free variable x defines a property of numbers: the property satisfied exactly by the numbers n such that $F(s^n(0))$ is true in the canonical structure.

We can easily construct formulas $Even(x)$, $Prime(x)$, $Power\_of\_two(x)$ ...

Via the encoding formulas "are" numbers, and so a formula also defines a property of formulas!

numbers $\longrightarrow$ formulas

formula $F(x)$ $\longrightarrow$ set of numbers $\longrightarrow$ set of formulas

Going further ...

We can (less easily) construct formulas like

- $First\_symbol\_is\_\forall(x)$
- $At\_least\_ten\_symbols(x)$
- $Closed(x)$
- ...

that are true i.t.c.s. for $x := s^n(0)$ iff the number n encodes a formula and the formula satisfies the corresponding property.

And even further ...

We can construct (even less easily) a formula

- $In\_Q(x)$

that is true i.t.c.s. for $x = s^n(0)$ iff the number n encodes a closed formula F such that $F \in$ Q.

The reason is

$$F \in \text{Q} \;\text{ iff }\; \text{Q1}, \ldots, \text{Q9} \models F \;\text{ iff }\; \text{Q1}, \ldots, \text{Q9} \vdash F$$

and the derivation procedure amounts to symbol manipulation.

Same for any other set $\mathcal{X}$ of axioms.

Diagonal Lemma

Recall our goal: Given a set of axioms $\mathcal{X} \subseteq \texttt{Arith}$, construct a formula $F \in \texttt{Arith}$ such that $F \notin T_{\mathcal{X}}$

Let $\underline{\texttt{F}}$ denote the term $s^n(0)$ where n is the Gödel encoding of the formula F.

Intuition: $\underline{\texttt{F}}$ is a "name" we give to F

Lemma (Diagonal Lemma): Let $\mathcal{X}$ be any set of axioms containing Q1, … Q9. For every formula $B(y)$ there is a closed formula G such that $G \leftrightarrow B(\underline{\texttt{G}}) \in T_{\mathcal{X}}$.

We call G the Gödel formula of $B(x)$.
We have: G true i.t.c.s if and only if G has property B

Intuition: G asserts that G has property B (true or false i.t.c.s.!)

Reaching the goal

Theorem: Let $\mathcal{X}$ be any set of axioms containing Q1, … Q9.
Let $G_{\mathcal{X}}$ be the Gödel formula of $\neg In\_T_{\mathcal{X}}(x)$. Then $G_{\mathcal{X}} \in \texttt{Arith} \setminus T_{\mathcal{X}}$.

Proof idea: By definition, $G_{\mathcal{X}}$ is true i.t.c.s iff $G_{\mathcal{X}} \notin T_{\mathcal{X}}$.

If $G_{\mathcal{X}}$ is false i.t.c.s. then $G_{\mathcal{X}} \in T_{\mathcal{X}}$.
Since $\mathcal{X} \subseteq \texttt{Arith}$, we have $G_{\mathcal{X}} \in \texttt{Arith}$.
But then, by definition of $\texttt{Arith}$, $G_{\mathcal{X}}$ is true i.t.c.s.
Contradiction!

So $G_{\mathcal{X}}$ is true i.t.c.s., i.e., $G_{\mathcal{X}} \in \texttt{Arith}$.
But then $G_{\mathcal{X}} \notin T_{\mathcal{X}}$, and so $G_{\mathcal{X}} \in \texttt{Arith} \setminus T_{\mathcal{X}}$. Done!

Gödel's second incompleteness theorem

For any set of axioms $\mathcal{X}$ containing Q1 we have $0 = s(0) \notin T_{\mathcal{X}}$, and so $T_{\mathcal{X}}$ is consistent iff $0 = s(0) \notin T_{\mathcal{X}}$.

The consistency formula for $\mathcal{X}$ is the formula $\neg In\_T_{\mathcal{X}}(\underline{\texttt{0=s(0)}})$

Intuition: The consistency formula for $\mathcal{X}$ states that $T_{\mathcal{X}}$ is consistent.

Theorem (Gödel's second incompleteness theorem): Let $\mathcal{X}$ be any set of axioms containing P. Then the consistency formula for $\mathcal{X}$ does not belong to $T_{\mathcal{X}}$.

Intuition: the consistency of a theory cannot be derived from the axioms of the theory.

Proving the Diagonal Lemma: Diagonalization

Let $F(x)$ be a formula with a free variable x.
The diagonalization of F is the closed formula

$$DiagF := \exists x \; x = \underline{\texttt{F}} \wedge F(x)$$

Intuition: $DiagF$ asserts that F has property F

Observe: $DiagF$ and $F(\underline{\texttt{F}})$ are logically equivalent, but they have different Gödel numbers.

The representation theorem

Theorem : There is a formula $Diag(x, y)$ such that for every formula F

$$\forall y \quad Diag(\underline{\texttt{F}}, y) \leftrightarrow y = \underline{\texttt{DiagF}}$$

can be derived in Q (and so in P).

Proof: Omitted.

Observe: the theorem does not hold for every set of axioms. For instance, it does not hold for the system Q1-Q4, since in that system we cannot infer anything about the product function.

Proof of the Diagonal Lemma I

Lemma: Let $\mathcal{X}$ be any set of axioms containing Q1, ...Q9.
For every formula $B(y)$ there is a closed formula G such that $G \leftrightarrow B(\underline{\texttt{G}}) \in T_{\mathcal{X}}$.

Proof: Let $A(x) := \exists y \; (Diag(x, y) \wedge B(y))$ and let $G := DiagA$.

Intuition: G asserts that the diagonalization of A (the formula asserting that A satisfies A) satisfies B.

Explicitely:

$$G := \exists x \; (x = \underline{\texttt{A}} \wedge A(x)) := \exists x \; (x = \underline{\texttt{A}} \wedge \exists y \; (Diag(x, y) \wedge B(y)))$$

Proof of the Diagonal Lemma II

The formula $G \leftrightarrow \exists y \; (Diag(\underline{\texttt{A}}, y) \wedge B(y))$ is valid, and so, since valid formulas belong to every theory, we have

$$G \leftrightarrow \exists y \; (Diag(\underline{\texttt{A}}, y) \wedge B(y)) \;\; \in \;\; T_{\mathcal{X}}$$

Since $G := DiagA$, we have by the representation theorem:

$$\forall y \; (Diag(\underline{\texttt{A}}, y) \leftrightarrow y = \underline{\texttt{G}}) \;\; \in \;\; T_{\mathcal{X}}$$

And so, since $T_{\mathcal{X}}$ is closed under consequence, we get

$$G \leftrightarrow \exists y \; (y = \underline{\texttt{G}} \; \wedge \; B(y)) \;\; \in \;\; T_{\mathcal{X}}$$

and for the same reason

$$G \leftrightarrow B(\underline{\texttt{G}}) \;\; \in \;\; T_{\mathcal{X}}$$