

Herbrand universe

The **Herbrand universe** $D(F)$ of a closed formula F in Skolem form is the set of all terms without variables that can be constructed using the function symbols and constants of F .

In the special case that F contains no constants, we first pick an arbitrary constant, say a , and then construct the variable-free terms.

Formally, $D(F)$ is inductively defined as follows:

- (1) All constants occurring in F belong to $D(F)$; if no constant occurs in F , then $a \in D(F)$.
- (2) For every n -ary function symbol f occurring in F , if $t_1, t_2, \dots, t_n \in D(F)$ then $f(t_1, t_2, \dots, t_n) \in D(F)$.

Herbrand structure

Let F be a closed formula in Skolem form. A structure $\mathcal{A} = (U_{\mathcal{A}}, I_{\mathcal{A}})$ suitable for f is a **Herbrand structure** for F if it satisfies the following conditions:

- (1) $U_{\mathcal{A}} = D(F)$, and
- (2) for every n -ary function symbol f occurring in F and every $t_1, t_2, \dots, t_n \in D(F)$: $f^{\mathcal{A}}(t_1, t_2, \dots, t_n) = f(t_1, t_2, \dots, t_n)$.

Fundamental theorem of predicate logic

Theorem: A closed formula F in Skolem form is satisfiable if and only if it has a Herbrand model.

Proof: If the formula has a Herbrand model then it is satisfiable.

For the other direction let $\mathcal{A} = (U_{\mathcal{A}}, I_{\mathcal{A}})$ be an **arbitrary** model of F .

We define a Herbrand structure $\mathcal{B} = (U_{\mathcal{B}}, I_{\mathcal{B}})$ as follows:

Universe $U_{\mathcal{B}} = D(F)$

Function symbols $f^{\mathcal{B}}(t_1, t_2, \dots, t_n) = f(t_1, t_2, \dots, t_n)$

Predicate symbols $(t_1, \dots, t_n) \in P^{\mathcal{B}}$ iff $(\mathcal{A}(t_1), \dots, \mathcal{A}(t_n)) \in P^{\mathcal{A}}$

Claim: \mathcal{B} is also a model of F .

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We prove a stronger assertion:

For every closed form G in Skolem form such that G^* only contains atomic formulas of F^* : if $\mathcal{A} \models G$ then $\mathcal{B} \models G$

Proof: By induction on the number n of universal quantifiers of G .

Basis ($n = 0$). Then G has no quantifiers at all.

It follows $\mathcal{A}(G) = \mathcal{B}(G)$, and we are done.

Step ($n > 0$). Then $G = \forall x H$.

$$\mathcal{A} \models G$$

$$\Rightarrow \text{for every } u \in U_{\mathcal{A}}: \mathcal{A}_{[x/u]}(H) = 1$$

$$\Rightarrow \text{for every } u \in U_{\mathcal{A}} \text{ of the form } u = t^{\mathcal{A}}$$

$$\text{where } t \in D(G): \mathcal{A}_{[x/u]}(H) = 1 \quad (\text{Skolem!})$$

$$\Rightarrow \text{for every } t \in D(G): \mathcal{A}_{[x/t^{\mathcal{A}}]}(H) = 1$$

$$\Rightarrow \text{for every } t \in D(G): \mathcal{A}(H[x/t]) = 1 \quad (\text{translation lemma})$$

$$\Rightarrow \text{for every } t \in D(G): \mathcal{B}(H[x/t]) = 1 \quad (\text{induction hypothesis})$$

$$\Rightarrow \text{for every } t \in D(G): \mathcal{B}_{[x/t^{\mathcal{B}}]}(H) = 1 \quad (\text{translation lemma})$$

$$\Rightarrow \text{for every } t \in D(G): \mathcal{B}[x/t](H) = 1 \quad (\mathcal{B} \text{ is Herbrand structure})$$

$$\Rightarrow \mathcal{B}(\forall x H) = 1 \quad (U_{\mathcal{B}} = D(G))$$

$$\Rightarrow \mathcal{B} \models G$$

Herbrand expansion

Let $F = \forall y_1 \forall y_2 \dots \forall y_n F^*$ be a closed formula in Skolem form. The **Herbrand expansion** of F is the set of atomic formulas

$$E(F) = \{F^*[y_1/t_1][y_2/t_2] \dots [y_n/t_n] \mid t_1, t_2, \dots, t_n \in D(F)\}$$

Informally: the formulas of $E(F)$ are the result of substituting the variables of F^* by the terms of $D(F)$ in every possible way.

Gödel-Herbrand-Skolem's Theorem

Theorem: A closed formula F in Skolem form is satisfiable if and only if its Herbrand expansion $E(F)$ is satisfiable (in the sense of propositional logic).

Proof: It suffices to show: if $E(F)$ is satisfiable, then F has a Herbrand model.

Let F be of the form $\forall y_1 \forall y_2 \dots \forall y_n F^*$. We have:

\mathcal{A} is a Herbrand model of F

iff for every $t_1, t_2, \dots, t_n \in D(F)$:

$$\mathcal{A}_{[y_1/t_1][y_2/t_2]\dots[y_n/t_n]}(F^*) = 1$$

iff for every $t_1, t_2, \dots, t_n \in D(F)$:

$$\mathcal{A}(F^*[y_1/t_1][y_2/t_2]\dots[y_n/t_n]) = 1$$

iff for every $G \in E(F)$ gilt $\mathcal{A}(G) = 1$

iff \mathcal{A} is a model of $E(F)$

Herbrand's Theorem

Theorem: A closed formula F in Skolem form is unsatisfiable if and only if some **finite subset** of the Herbrand expansion of $E(F)$ is unsatisfiable.

Proof: Follows immediately from the Gödel-Herbrand-Skolem's Theorem and the Compactness Theorem.

Gilmore's Algorithm

Let F be closed formula in Skolem form and let $\{F_1, F_2, F_3, \dots, \}$ be an enumeration of $E(F)$.

Input: F

$n := 0$;

repeat $n := n + 1$;

until $(F_1 \wedge F_2 \wedge \dots \wedge F_n)$ is unsatisfiable;

report "unsatisfiable" and **halt**.

Löwenheim-Skolem's Theorem

Theorem: Every satisfiable formula of predicate logic has a model with a countable universe.

Proof: Let F be a formula, and let G be a sat-equivalent formula in Skolem form. Then for every set X :

F has a model with universe X
iff
 G has a model with universe X .

F satisfiable \Rightarrow G satisfiable
 \Rightarrow G has a Herbrand model (X, I_1)
 \Rightarrow F has a model (X, I_2)
 \Rightarrow F has a countable model
(Herbrand universes are countable)