Binary Decision Diagrams

Binary Decision Diagrams (BDDs) are a class of graphs that can be used as data structure for compactly representing boolean functions.

BDDs were introduced by R. Bryant in 1986.

BDDs are used to solve equivalence problems between formulas of propositional logic.

Very important in the areas of hardware design and hardware optimization.

Graphs

Recall some basic graph-theoretical concepts:

directed graph	node	edge	
predecessor	successor	path	cycle
acyclic graph	tree	forest	

Boolean Functions

A boolean function of arity $n \geq 1$ is a function $\{0,1\}^n \to \{0,1\}$.

Examples:

$$\mathrm{or}(x_1, x_2) = \begin{cases} 1 & \text{if } x_1 = 1 \text{ or } x_2 = 1 \\ 0 & \text{if } x_1 = 0 \text{ or } x_2 = 0 \end{cases}$$

$$\text{if\_then\_else}(x_1, x_2, x_3) = \begin{cases} x_2 & \text{if } x_1 = 1 \\ x_3 & \text{if } x_1 = 0 \end{cases}$$

Z.B.: if_then_else$(1,0,1) = 0$, if_then_else$(0,0,1) = 1$

$$\text{sum}(x_1, x_2, x_3, x_4) = \begin{cases} 1 & \text{if } x_1 + x_2 = x_3 x_4 \\ 0 & \text{otherwise} \end{cases}$$

Z.B.: sum$(1,1,1,0) = 1$ (because $1 + 1 = 10$), sum$(0,0,0,1) = 0$ (because $0 + 0 = 00$).

$$\text{majority}_n(x_1,\ldots,x_n) = \begin{cases} 1 & \text{if the majority of } x_1,\ldots,x_n \text{ has value } 1 \\ 0 & \text{otherwise} \end{cases}$$

Z.B.: $\text{majority}_4(1,1,0,0) = 0$, $\text{majority}_3(1,0,1) = 1$

$$\text{parity}_n(x_1,\ldots,x_n) = \begin{cases} 1 & \text{if the number of inputs } x_1,\ldots,x_n \text{ equal to } 1 \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$

Z.B.: $\text{parity}_3(1,0,1) = 1$, $\text{parity}_2(1,0) = 0$

Let F be a formula, and let n be a number such that all atomic formulas occurring in F belong to $\{A_1,\ldots,A_n\}$.

Example: $F = A_1 \wedge A_2$, $n = 2$, but also $n = 3$!

We define the boolean function $f_F^n\colon \{0,1\}^n \to \{0,1\}$:

$$f_F^n(x_1,\ldots,x_n) = \text{truth value of } F \text{ under the assignment that sets } A_1,\ldots,A_n \text{ to } x_1,\ldots,x_n$$

Example: For $F = A_1 \wedge A_2$:

$$\begin{aligned} f_F^2(0,1) &= \text{value of } 0 \wedge 1 = 0 \\ f_F^3(0,1,1) &= \text{value of } 0 \wedge 1 = 0 \end{aligned}$$

Remark: If all of $\{A_1,\ldots,A_n\}$ occur in F, then f_F^n is essentially the truth table of F.

Convention: We write e.g. $f(x_1,x_2,x_3) = x_1 \vee (x_2 \wedge \neg x_1)$, meaning $f = f_F^3$ for the formula $F = A_1 \vee (A_2 \wedge \neg A_1)$.

Fact: Let F_1 and F_2 be two formulas, and let n be a number such that all atomic formulas occurring in F_1 or F_2 belong to $\{A_1,\ldots,A_n\}$. Then $f_{F_1}^n = f_{F_2}^n$ iff $F_1 \equiv F_2$.

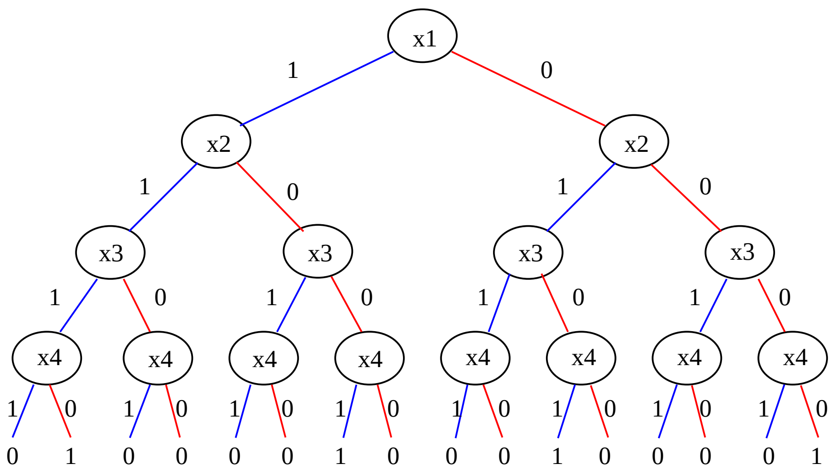
Example: $F_1 = A_1$, $F_2 = A_1 \wedge (A_2 \vee \neg A_2)$.

$$\begin{aligned} f_{F_1}^2(0,0) &= 0 = f_{F_2}^2(0,0) \\ f_{F_1}^2(0,1) &= 0 = f_{F_2}^2(0,1) \\ f_{F_1}^2(1,0) &= 1 = f_{F_2}^2(1,0) \\ f_{F_1}^2(1,1) &= 1 = f_{F_2}^2(1,1) \end{aligned}$$

Convention: The constants $\mathbf{0}$ and $\mathbf{1}$ represent the only two boolean functions of arity 0.

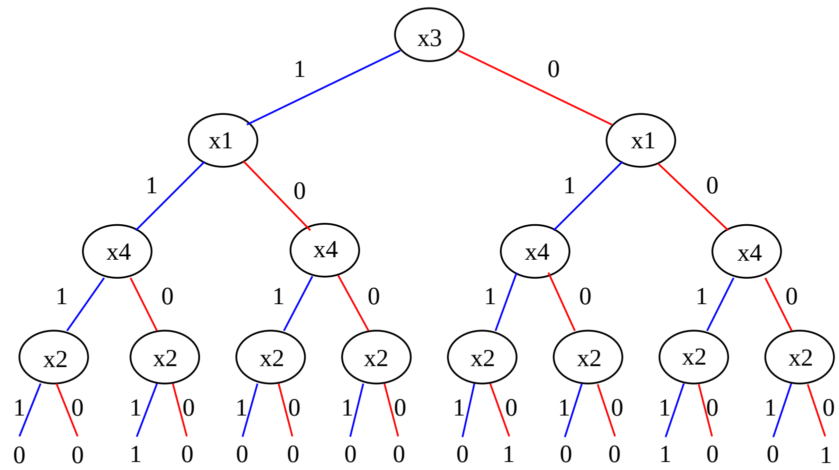
sum as binary decision tree

A boolean function can be represented by a decision tree



Variable order

A decision tree can use a variable order different from the order used in the function.



A variable order is a bijection

$$b\colon \{1,\ldots,n\} \to \{x_1,\ldots,x_n\}$$

We say that $b(1), b(2), b(3), \ldots, b(n)$ are the first, second, third, ..., n-th variable w.r.t. the order b.

We denote the bijection $b(1) = x_{i_1}, \ldots, b(n) = x_{i_n}$ by

$$x_{i_1} < x_{i_2} < \ldots < x_{i_n} \; .$$

Binary decision trees

A decision tree for the variable order $x_{i_1} < \ldots < x_{i_n}$ is a tree satisfying the following conditions:

(1) All leaves are labelled by 0 or by 1.

(2) All other nodes are labelled by a variable and have exactly two children, the 0-child and the 1-child. The edges leading to these children are labelled by 0 resp. by 1.

(3) If the root is not a leave, then it is labelled by x_{i_1}.

(4) If a node is labelled by x_{i_n} then its two children are leaves.

(5) If a node is labelled by x_{i_j} and $j < n$, then its two children are labelled by $x_{i_{j+1}}$.

Every path of a decision tree determines an assignment of the variables $x_{i_1}, \ldots x_{i_n}$ and vice versa.

The boolean function f_T represented by a decision tree T is defined as follows:

$f_T(x_1, \ldots, x_n) =$ label of the leaf reached by the path corresponding to the assignment $x_{i_1}\, x_{i_2}\, \ldots\, x_{i_n}$

A binary decision forest ist a forest of decision trees with the same variable order. A decision forest represents the set of functions represented by its elements.

Binary Decision Diagrams (informally)

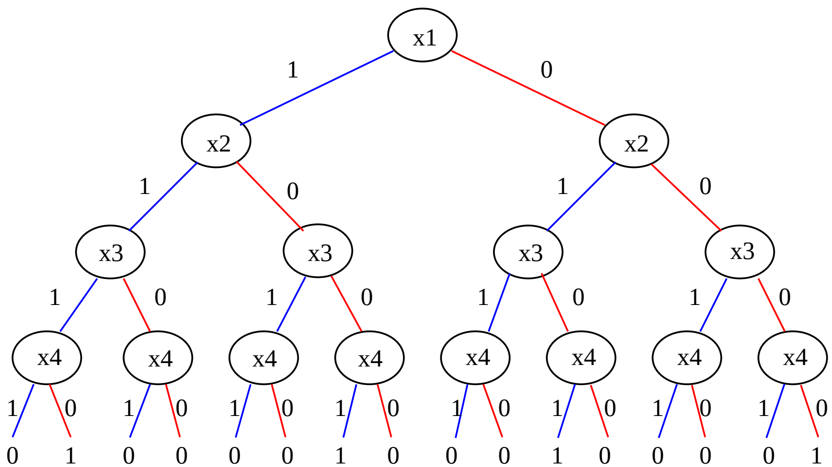
A BDD (multiBDD) is a "compact representation" of a binary decision tree (decision forest).

A BDD (multiBDD) is obtained from a decision tree (forest) through repeated application of two compression rules (see example in the next slide):

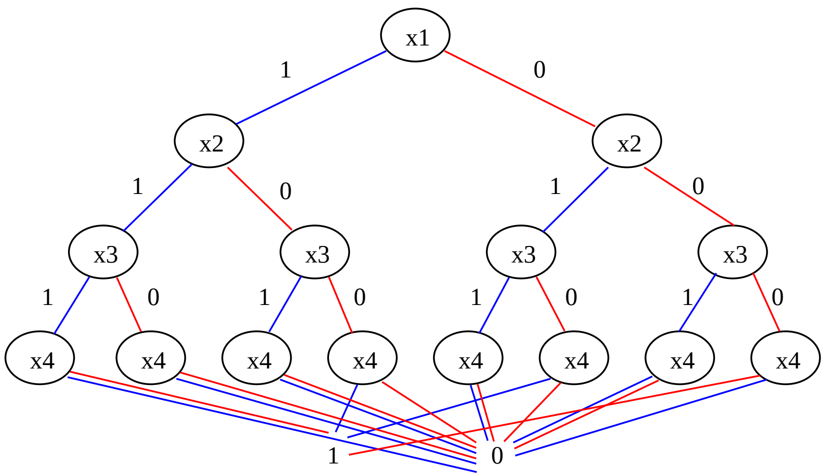
- Rule 1: Sharing of identical subtrees.
- Rule 2: Elimination of nodes for which the 0-child and the 1-child coincide (redundant nodes).

The rules are applied until all subtrees are different and there are no redundant nodes.

Example: sharing of subtrees

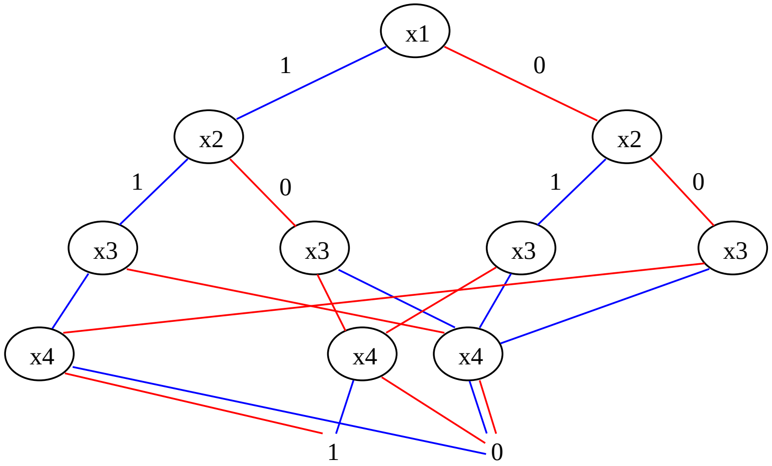


Example: sharing of subtrees



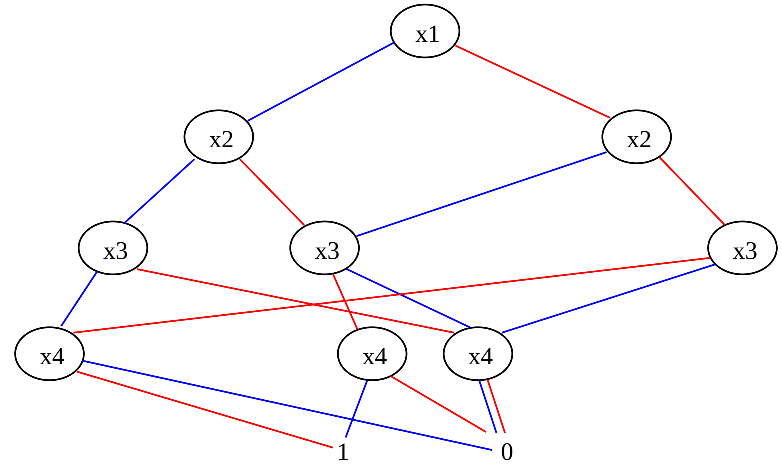
All 0- und 1-leaves are merged.

Example: sharing of subtrees



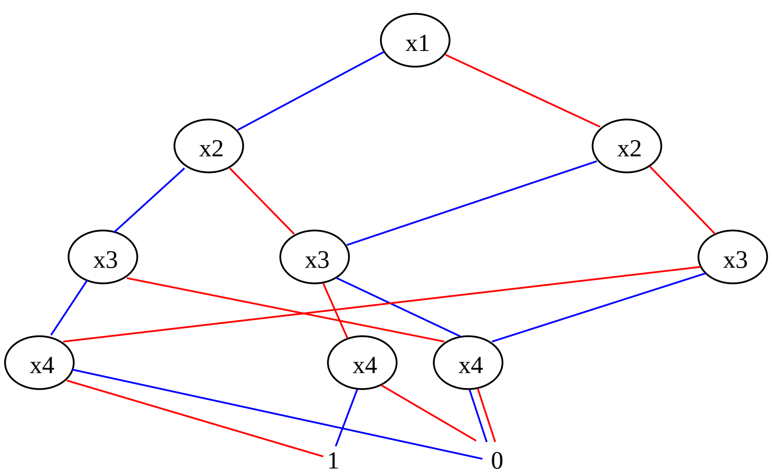
Identical x_4-nodes are merged.

Example: sharing of subtrees

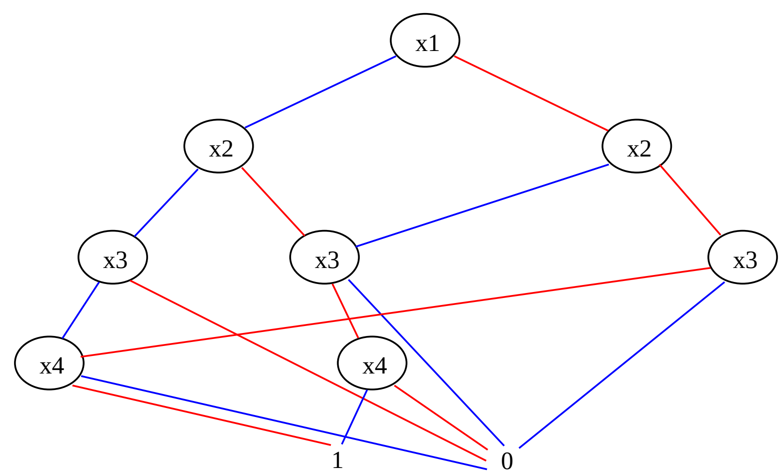


Identical x_3-nodes are merged.

Example: removing redundant nodes



Example: removing redundant nodes



Redundant x_4-node is removed

Formal Definition of BDDs

A BDD for a given variable order is an acyclic directed graph satisfying the following properties:

(1) There is exactly one node without predecessors (the root)

(2) There is one or two nodes without succesors, labelled by 0 or 1 (if there are two then they carry different labels).

(3) All other nodes are labelled by a variable and have exactly two distinct children, the 0-child and the 1-child. The edges leading to these children are labelled by 0 resp. by 1.

(4) A child of a node is labelled by 0, by 1, or by a variable larger than the label of its parent w.r.t. the variable order.

(5) All descendant-closed subgraphs of the graph are non-isomorphic.

MultiBDDs

A multiBDD is an acyclic graph satisfying (2)-(5) together a distinguished nonempty subset of nodes called the roots.

Every node without predecessors is a root, but other nodes may also be roots.

A multiBDD represents a set of boolean functions, one for each root.

Remarks

Remark: A "closed subgraph" of a BDD is again a BDD.

Remark: The function true$_n(x_1, \ldots, x_n)$ given by

$$\text{true}_n(x_1, \ldots, x_n) = 1 \text{ for every } x_1, x_n \in \{0, 1\}^n$$

is represented, for every $n \geq 1$ and for every variable order, by the BDD consisting of one single node labelled by 1.

Similarly for false$_n(x_1, \ldots, x_n)$

Relevance of variable orders

The variable order can have large impact on the size of the BDD.

Example:

$$f(x_1, \ldots, x_{2n}) = (x_1 \leftrightarrow x_{n+1}) \wedge (x_2 \leftrightarrow x_{n+2}) \wedge \cdots \wedge (x_n \leftrightarrow x_{2n})$$

Size grows exponentially in n for $x_1 < \cdots < x_n < x_{n+1} < \cdots < x_{2n}$.
Size grows linearly in n for $x_1 < x_{n+1} < x_2 < x_{n+2} < \ldots < x_n < x_{2n}$.

Problem in practice: finding a good order.

Canonicity of BDDs

We show that for a given boolean function and a given variable order there is a unique BDD representing the function.

More generally (but simpler to prove!), we show that for every set of boolean functions of the same arity and for every variable order there is a unique multiBDD representing the set.

The functions $f[0]$ und $f[1]$

Lemma I: Let f be a boolean function of arity $n \geq 1$. There are exactly two boolean functions $f[0]$ und $f[1]$ of arity $(n-1)$ satisfying

$$f(x_1, \ldots, x_n) = (\neg x_1 \wedge f[0](x_2, \ldots, x_n)) \vee (x_1 \wedge f[1](x_2, \ldots, x_n)) \qquad (*)$$

Proof: The functions $f[0]$ and $f[1]$ defined by
$f[0](x_2, \ldots, x_n) = f(0, x_2, \ldots, x_n)$ and
$f[1](x_2, \ldots, x_n) = f(1, x_2, \ldots, x_n)$ satisfy (*).

Let f_0 and f_1 be arbitrary functions satisfying (*). Then
$f(x_1, \ldots, x_n) = (\neg x_1 \wedge f_0(x_2, \ldots, x_n)) \vee (x_1 \wedge f_1(x_2, \ldots, x_n))$

By the properties of $\vee$ and $\wedge$ we have
$f(0, x_2, \ldots, x_n) = f_0(x_2, \ldots, x_n)$ and together with
$f(0, x_2, \ldots, x_n) = f[0](x_2, \ldots, x_n)$ we get $f_0 = f[0]$.

The proof that $f_1 = f[1]$ holds is analogous.

Let $f\colon \{0,1\}^n \to \{0,1\}$ be a boolean function, let B be a BDD with variable order $x_1 < x_2 < \ldots < x_n$, and let v be the root of B. Define the nodes $v[0]$ and $v[1]$ as follows:

(1) If v is labelled by x_1, then $v[0]$ and $v[1]$ are the 0-child and the 1-child of v.

(2) Otherwise, $v[0] = v = v[1]$.

Lemma II: B represents the function f iff $v[0]$ and $v[1]$ represent the functions $f[0]$ and $f[1]$, respectively.
Proof: Easy.

Theorem: Let $\mathcal{F}$ be a nonempty set of boolean functions of arity n and let $x_{i_1} < \ldots < x_{i_n}$ be a variable order. There is exactly one multiBDD that follows this order and represents $\mathcal{F}$.

Proof: We consider the order $x_1 < x_2 < \ldots < x_n$, for other orders the proof is similar. Proof by induction on the arity n.
Basis: $n = 0$. There are exactly two boolean functions with $n = 0$, namely the constants $\mathbf{0}$ and $\mathbf{1}$, and two BDDs $\mathbf{K_0}, \mathbf{K_1}$ consisting of one single node labelled by 0 or by 1. The set $\{\mathbf{0}\}$ is represented by $\mathbf{K_0}$, the set $\{\mathbf{1}\}$ by $\mathbf{K_1}$, and the set $\{\mathbf{0}, \mathbf{1}\}$ by the multiBDD consisting of $\mathbf{K_0}$ and $\mathbf{K_1}$.

Step: $n > 0$. Let $\mathcal{F} = \{f_1, \ldots, f_k\}$.

Define $\mathcal{F}' = \{f_1[0], f_1[1], \ldots, f_k[0], f_k[1]\}$, where $f_i[0]$ and $f_i[1]$ are as in Lemma I.

By induction hypothesis there is exactly one multiBDD B' with roots $v_{10}, v_{11}, \ldots, v_{k0}, v_{k1}$ representing $\mathcal{F}'$. I.e., for every function $f_i[j]$ the root v_{ij} represents $f_i[j]$.

Let B be the multiBDD with roots $v_1, \ldots v_n$ obtained from B' after executing the following steps for $i = 1, 2, \ldots, k$:

- If $v_{i0} = v_{i1}$ then set $v_i := v_{i0}$.
  (In this case v_{i0} represents f_i.)
- If $v_{i0} \neq v_{i1}$ and B' has a node v such with v_{i0} as 0-child and v_{i1} as 1-child then set $v_i := v$.
- If $v_{i0} \neq v_{i1}$ and B' contains no such node then add a new node v_i having v_{i0} as 0-Kind and v_{i1} as 1-Kind .
  (So v_i represents f_i, see Lemma II.)

Clearly, B represents $\mathcal{F}$. We now show that B is the only multiBDD representing $\mathcal{F}$.

Let $\tilde{B}$ be an arbitrary multiBDD with roots $\tilde{v}_1, \ldots \tilde{v}_n$ representing $\mathcal{F}$.

By Lemma II, $\tilde{B}$ contains nodes $\tilde{v}_1[0], \tilde{v}_1[1], \ldots, \tilde{v}_k[0], \tilde{v}_k[1]$ representing the functions of $\mathcal{F}'$.

By induction hypothesis, the multiBDD containing these nodes and all its descendants is the multiBDD B'. In particular, we have $v_{ij} = \tilde{v}_i[j]$ for every $i \in \{1, \ldots, k\}$ and $j \in \{0, 1\}$.

Let v_i and $\tilde{v}_i$ be the roots of B and $\tilde{B}$, representing f_i. By Lemmas I und II, v_{i0} and $\tilde{v}_i[0]$ represent $f[0]$, and v_{i1} and $\tilde{v}_i[1]$ represent $f[1]$.

Since $v_i[0] = \tilde{v}_i[0]$ and $v_i[1] = \tilde{v}_i[1]$ we get $v_i = \tilde{v}_i$.

So B and $\tilde{B}$ are equal.

Computing BDDs from Formulas

Goal: Given a formula F over the atomic formulas $A_1, \ldots, A_n$ and a variable order for $\{x_1, \ldots, x_n\}$, compute a BDD representing $f_F(x_1, \ldots, x_n)$.

Naive procedure: Compute the decision tree of f_F and reduce it using the compression rules.

Problem: The decision tree is too large!

Better procedure (idea): Compute recursively the multiBDD representing $\{f_{F[A_i/0]}, f_{F[A_i/1]}\}$ for a suitable A_i, and derive from it the BDD for f_F, where $F[A_i/0]$ bzw. $F[A_i/0]$ are the formulas obtained by replacing every occurrence of A_i by 0 resp. by 1.

In the next slides we formalize this idea.

Let $\mathcal{S} = \{F_1, \ldots, F_n\}$ be a nonempty set of formulas.

We define a procedure multiBDD($\mathcal{S}$) that returns the roots of a multiBDD representing the set $\{f_{F_1}, \ldots, f_{F_n}\}$.

$\mathbf{K_0}$ denotes the BDD with only one node labelled by 0.
$\mathbf{K_1}$ denotes the BDD with only one node labelled by 1.

A proper formula is a formula containing at least one occurrence of a variable (i.e., not only 0 and 1).

An atomic formula A_i is smaller than A_j if x_i appears before x_j in the variable order.

The function multiBDD($\mathcal{S}$)

if $\mathcal{S}$ contains no proper formulas
then if all formulas of $\mathcal{S}$ are equivalent to 0
    then return $\{\mathbf{K_0}\}$
    else if all formulas in $\mathcal{S}$ are equivalent to 1
    then return $\{\mathbf{K_1}\}$
    else return $\{\mathbf{K_0}, \mathbf{K_1}\}$
else choose a proper formula $F \in \mathcal{S}$.
    Let A_i be the smallest atomic formula occurring in F.
    Let $B =$ multiBDD($(\mathcal{S} \setminus \{F\}) \cup \{F[A_i/0], F[A_i/1]\}$).
    Let v_0, v_1 be the roots of B representing $F[A_i/0], F[A_i/1]$.
    if $v_0 = v_1$ **then return** B
    else add a new node v with v_0, v_1 as 0- and 1-child
        (if such a node does not exist yet);
        return $(B \setminus \{v_0, v_1\}) \cup \{v\}$

Equivalence problems

Given two formulas F_1, F_2, the following algorithm decides whether $F_1 \equiv F_2$ holds:

- Choose a suitable variable order $x_1 < \ldots < x_n$.
- Compute a multiBDD for $\{F_1, F_2\}$.
- Check whether the roots v_{F_1}, v_{F_2} are equal.

For digital circuits: the BDDs are not derived from formulas, but directly from the circuits.

Operations on BDDs

Given:

- two formulas F, G over the atomic formulas $A_1, \ldots, A_n$,
- a variable order for $\{x_1, \ldots, x_n\}$,
- a multiBDD with two roots v_F, v_G representing the functions $f_F(x_1, \ldots, x_n)$ and $f_F(x_1, \ldots, x_n)$, and
- a binary boolean operation (e.g. $\vee, \wedge, \rightarrow, \leftrightarrow$)

Goal: compute a BDD for the function $f_{F \circ G}(x_1, \ldots, x_n)$.

With our convention we have $f_{F \circ G} = f_F \circ f_G$

Idea

Lemma: $(f_F \circ f_G)[0] = f_F[0] \circ f_G[0]$ and $(f_F \circ f_G)[1] = f_F[1] \circ f_G[1]$.

Proof: Exercise.

Algorithm: (for the order $x_1 < x_2 < \ldots < x_n$, similar for others)

- Compute a multiBDD for $\{f_F[0] \circ f_G[0], f_F[1] \circ f_G[1]\}$. (Recursively.)
- Use the Lemma to build a BDD for $f_{F \circ G}(x_1, \ldots, x_n)$.

The function Or(v_F, v_G)

if $v_F = \mathbf{K_1}$ or $v_F = \mathbf{K_1}$ **then return** $\mathbf{K_1}$
else if $v_F = v_G = \mathbf{K_0}$ **then return** $\mathbf{K_0}$
else let v_{F0}, v_{G0} be the nodes for $F[0], G[0]$ and
  let v_{F1}, v_{G1} be the nodes for $F[1], G[1]$
  $v_0 :=$ Or(v_{F0}, v_{G0}); $v_1 :=$ Or(v_{F1}, v_{G1})
  if $v_0 = v_1$ **then return** v_0
  else add a new node v with v_0, v_1 as 0- and 1-child
    (if such a node does not exist yet);
    return v

Implementing BDDs

Source: *An introduction to Binary Decision Diagrams*
Prof. H.R. Andersen
http://www.itu.dk/people/hra/notes-index.html

Data structures

BDD-nodes coded as numbers $0, 1, 2, \ldots$ with 0, 1 for the end nodes.

BDD-nodes are stored in a table

$$T\colon u \mapsto (i, l, h)$$

where i, l, h are the label, the 0-child and the 1-child of u.
(Here l stands for "low" and h for "high".)

We maintain a second table

$$H\colon (i, l, h) \mapsto u$$

so that following invarinat holds:

$$T(u) = (i, l, h) \quad \text{iff} \quad H(i, l, h) = u$$

Basic operations on T:

$init(T)$:	Initializes T with 0 and 1
$add(T, i, l, h)$:	Adds node with attributes (i, l, h) to T and returns it
$var(u)$, $low(u)$, $high(u)$:	Returns the variable, 0-child, 1-child of u

Basic operations on H:

$init(H)$:	Initializes H as the empty table
$member(H, i, l, h)$:	Checks whether (i, l, h) belongs to H
$lookup(H, i, l, h)$:	Returns the node $H(i, l, h)$
$insert(H, i, l, h, u)$:	Adds $(i, l, h) \mapsto u$ to H (if not yet there)

The function $Make(i, l, h)$

Look in H for a node with atributes (i, l, h). If found, then return it. Otherwise create a new node and return it.

$Make(i, l, h)$

```
if l = h then return l
else if member(H, i, l, h) then
    return lookup(H, i, h, l)
else u := add(T, i, l, h)
    insert(H, i, l, h, u)
    return u
```


Implementing Or

Problem: the function can be called many times with the same arguments.

Solution: dynamic programming. The results of all calls are stored. Each call checks first if the result has already been computed earlier.

$Or(u_1, u_2)$

```
init G
return Or'(u1, u2)
```


$Or'(u_1, u_2)$

```
if G(u1, u2) ≠ empty then return G(u1, u2)
else if u1 = 1 or u2 = 1 then return 1
else if u1 = 0 and u2 = 0 then return 0
else if var(u1) = var(u2) then
    u := Make(var(u1), Or'(low(u1), low(u2)),
                       Or'(high(u1), high(u2)))
else if var(u1) < var(u2) then
    u := Make(var(u1), Or'(low(u1), u2), Or'(high(u1), u2))
else u := Make(var(u2), Or'(u1, low(u2)), Or'(u1, high(u2)))
G(u1, u2) = u
return u
```