Technische Universität München **Summer term 2011**
Theoretische Informatik
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Logic – Endterm 2

***Please note*: If not stated otherwise, all answers have to be justified.**

Exercise 1 **2P+1P+1P=4P**

Given is the following formula F:

$$(D \vee \neg E) \wedge (\neg B \vee \neg E \vee C) \wedge (\neg A \vee B) \wedge A \wedge \neg E.$$

(a) Decide whether F is satisfiable or not using the algorithm for Horn formulas discussed in the lecture.

(b) How many models defined only on A, B, C, D, E does F have?

(c) How many models does F have?

Exercise 2 **2P+2P+3P+3P=10P**

For this exercise, we introduce a *restricted Hilbert calculus* in which the set of axioms is restricted to:

Ax1: $(\neg F \rightarrow \neg G) \rightarrow (G \rightarrow F)$

Ax2: $F \rightarrow (\neg F \rightarrow G)$

(a) Consider the two (erroneous?) derivations below.

For each step, state whether it is correct or not in this restricted Hilbert calculus; if it is correct, explain why.

i)
1. $\{\neg A\} \vdash \neg A$
2. $\{\neg A\} \vdash \neg A \rightarrow (A \rightarrow B)$
3. $\{\neg A\} \vdash A \rightarrow B$

ii)
1. $\{A \rightarrow B\} \vdash (A \rightarrow B) \rightarrow (\neg(A \rightarrow B) \rightarrow (\neg A \rightarrow \neg B))$
2. $\{A \rightarrow B\} \vdash A \rightarrow B$
3. $\{A \rightarrow B\} \vdash \neg(A \rightarrow B) \rightarrow (\neg A \rightarrow \neg B)$.

(b) In each case, give a derivation in this restricted Hilbert calculus of the stated formula under the stated hypotheses:

i) $\{B\} \vdash A \rightarrow B$

ii) $\{A, A \rightarrow B, B \rightarrow C\} \vdash A \rightarrow C$

Exercise 3 **3P+3P=6P**

Let $\equiv_s$ denote equivalence up to satisfiability (equisatisfiability). Show each of the following equivalences: transform the left-hand side step-by-step into the right-hand side. **Clearly state in each step, how you transformed the formula and if equivalence or only equisatisfiability holds**.

(a) $\forall x \exists y \forall z \exists w (\neg P(a, w) \vee Q(f(x), y)) \equiv_s \forall x \forall z (\neg P(a, b) \vee Q(f(x), g(x)))$

(b) $\neg Q(z) \vee \neg \exists x R(x, y) \vee \forall x \exists y P(x, g(y, f(x))) \equiv_s \forall u \forall x (\neg Q(j(u, x)) \vee \neg R(x, i(u)) \vee P(u, g(h(u), f(u))))$

Exercise 4 **3P**

Use resolution with unification to derive the empty clause from the following first-order formula F in clause normal form:

$$\{\ \{\neg P(f(y_1)), Q(y_1, h(z_1, z_1))\},\ \{\neg P(f(f(x_2))), \neg Q(f(x_2), y_2)\},$$
$$\{P(f(x_3)), Q(x_3, h(y_3, a))\},\ \{\neg Q(f(y_4), z_4), \neg Q(f(a), h(f(a), y_5))\}\ \}$$

In each step, clearly state (i) which variables are renamed before the computation of a most general unificator, (ii) which literals are unified, and (iii) which most general unificator is used for the resolution step.

Exercise 5 **2P+3P+3P=8P**

Syllogisms have been introduced at the beginning of the lecture as an example of logical inference. In terms of first order logic, a syllogism consists of three formulas F_1, F_2, F_3 – two premises F_1, F_2, and a conclusion F_3 – where each formula takes the form of one of the following formulas up to renaming the predicate symbols P, Q:

$$(1) \quad \forall x(P(x) \to Q(x)) \quad (2) \quad \forall x(P(x) \to \neg Q(x)) \quad (3) \quad \exists x(P(x) \wedge Q(x)) \quad (4) \quad \exists x(P(x) \wedge \neg Q(x)).$$

A syllogism F_1, F_2, F_3 is valid if $\models (F_1 \wedge F_2) \to F_3$; otherwise the syllogism is not valid.

Example: In case of the syllogism "If all men are mortal, and Socrates is a man, then Socrates is mortal" we have

$$F_1 = \forall x(\text{man}(x) \to \text{mortal}(x)),\ F_2 = \exists x(\text{Socrates}(x) \wedge \text{man}(x))), \quad \text{and } F_3 = \exists x(\text{Socrates}(x) \wedge \text{mortal}(x))$$

In this example, F_1 is of the form (1), while F_2, F_3 are both of form (3).

(a) Give an example of a syllogism which is not valid. Prove your answer correct.

(b) Give an example of a syllogism which is valid where (i) $F_1 \wedge F_2$ has to be satisfiable, (iii) F_1, F_2, F_3 have to be pairwise distinct formulas, and (iii) at least one formula of F_1, F_2, F_3 has to be of form (4).

Prove the correctness of your answer using resolution.

(c) Describe an algorithm that, on input a syllogism $(F_1 \wedge F_2) \to F_3$, always terminates and correctly outputs whether the syllogism is valid or not; if it is not valid, your algorithm should also output a suitable structure $\mathcal{A}$ with $\mathcal{A} \not\models (F_1 \wedge F_2) \to F_3$.

Hint: Recall that Gilmore's algorithm terminates if the Herbrand universe is finite.

Exercise 6 **4P**

For a propositional variable A and a propositional formula F, let $F[A/b]$ denote the propositional formula obtained from F by substituting the boolean value b for each occurrence of A in F – if A does not occur in F at all then $F[A/b] = F$.

Let F, H be propositional formulas. Assume $\models (H[A/0] \leftrightarrow H[A/1])$ and $\models H \to F$.

Show that also $\models H \to (F[A/0] \leftrightarrow F[A/1])$.

Remark: Let $\mathcal{A}$ be an assignment, A a propositional variable, and $b \in \{0,1\}$. Recall that $\mathcal{A}_{[A/b]}$ is the assignment with $\mathcal{A}_{[A/b]}(A) = b$ and $\mathcal{A}_{[A/b]}(B) = \mathcal{A}(B)$ for any propositional variable B distinct from A.

Start from a satisfying assignment $\mathcal{A}$ of H (i.e. $\mathcal{A}(H) = 1$), and use that $\mathcal{A}_{[A/b]}(G) = \mathcal{A}(G[A/b])$ for every $b \in \{0,1\}$ and every propositional formula G.

Exercise 7 **2P+3P=5P**

(a) Let F be a first-order forumla in RPF, and G the formula obtained by Skolemizing F. It was shown in the lecture that any model $\mathcal{A}$ of G is also a model of F.

Show this result explicitly for the special case of $F = \forall x \exists y P(x,y)$ and $G = \forall x P(x, f(x))$.

(b) Show that any satisfiable formula F has an infinite model.

Hint: When exactly does the Herbrand universe $D(G)$ consist of infinitely many elements? For the case that $D(G)$ is finite, recall that, if $\mathcal{A} \models G \wedge H$, then also $\mathcal{A} \models G$.