Technische Universität München **Summer term 2011**
Theoretische Informatik
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SOLUTION

Logic – Endterm 2

***Please note*: If not stated otherwise, all answers have to be justified.**

Exercise 1 **2P+1P+1P=4P**

Given is the following formula F:

$$(D \vee \neg E) \wedge (\neg B \vee \neg E \vee C) \wedge (\neg A \vee B) \wedge A \wedge \neg E.$$

(a) Decide whether F is satisfiable or not using the algorithm for Horn formulas discussed in the lecture.

(b) How many models defined only on A, B, C, D, E does F have?

(c) How many models does F have?

Solution:

(a) In the first round A is marked being the only clause consisting of a single positive literal; subsequently, B is marked in the second round, after which the algorithm terminates with the result that $A = 1, B = 1, C = 0, D = 0, E = 0$ is a satisfying assignment for F.

(b) Necessarily, $A = 1, B = 1, E = 0$ which already satisfies all clauses of F. So C, D can be chosen arbitrarily. Hence, there are four minimal models.

(c) Infinitely many.

Exercise 2 **2P+2P+3P+3P=10P**

For this exercise, we introduce a *restricted Hilbert calculus* in which the set of axioms is restricted to:

Ax1: $(\neg F \to \neg G) \to (G \to F)$

Ax2: $F \to (\neg F \to G)$

(a) Consider the two (erroneous?) derivations below.

For each step, state whether it is correct or not in this restricted Hilbert calculus; if it is correct, explain why.

i)
1. $\{\neg A\} \vdash \neg A$
2. $\{\neg A\} \vdash \neg A \to (A \to B)$
3. $\{\neg A\} \vdash A \to B$

ii)
1. $\{A \to B\} \vdash (A \to B) \to (\neg(A \to B) \to (\neg A \to \neg B))$
2. $\{A \to B\} \vdash A \to B$
3. $\{A \to B\} \vdash \neg(A \to B) \to (\neg A \to \neg B)$.

(b) In each case, give a derivation in this restricted Hilbert calculus of the stated formula under the stated hypotheses:

i) $\{B\} \vdash A \to B$

ii) $\{A, A \to B, B \to C\} \vdash A \to C$

Solution:

(a) i) 1. Hypothesis

2. Neither an instance of Ax1 or Ax2, nor a hypothesis, nor obtainable from 1. using modus ponens, hence, not correct in the Hilbert calculus

(3. Modus ponens applied to 1. and 2.)

ii) 1. Ax2

2. Hypothesis

3. Modus ponens

(b) i)

$$\dfrac{\dfrac{\dfrac{}{\{B\} \vdash B}\,\{B\} \qquad \dfrac{}{\{B\} \vdash B \to (\neg B \to \neg A)}\,\text{Ax2}}{\{B\} \vdash \neg B \to \neg A} \qquad \dfrac{}{\{B\} \vdash (\neg B \to \neg A) \to (A \to B)}\,\text{Ax1}}{A \to B}$$

ii) Let $\Gamma = \{A, A \to B, B \to C\}$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash A}\,\Gamma \qquad \dfrac{}{\Gamma \vdash A \to B}\,\Gamma}{\Gamma \vdash B} \qquad \dfrac{}{\Gamma \vdash B \to C}\,\Gamma}{\Gamma \vdash C} \qquad \dfrac{}{\Gamma \vdash C \to (\neg C \to \neg A)}\,\text{Ax1}}{\Gamma \vdash \neg C \to \neg A} \qquad \dfrac{}{\Gamma \vdash (\neg C \to \neg A) \to (A \to C)}\,\text{Ax2}}{\Gamma \vdash A \to C}$$

Exercise 3 **3P+3P=6P**

Let $\equiv_s$ denote equivalence up to satisfiability (equisatisfiability). Show each of the following equivalences: transform the left-hand side step-by-step into the right-hand side. **Clearly state in each step, how you transformed the formula and if equivalence or only equisatisfiability holds**.

$$\begin{array}{lll} (a) & \forall x \exists y \forall z \exists w(\neg P(a, w) \lor Q(f(x), y)) & \equiv_s \quad \forall x \forall z(\neg P(a, b) \lor Q(f(x), g(x))) \\ (b) & \neg Q(z) \lor \neg \exists x R(x, y) \lor \forall x \exists y P(x, g(y, f(x))) & \equiv_s \quad \forall u \forall x(\neg Q(j(u, x)) \lor \neg R(x, i(u)) \lor P(u, g(h(u), f(u)))) \end{array}$$

Solution:

$$\begin{array}{lll} (a) & \forall x \exists y \forall z \exists w(\neg P(a, w) \lor Q(f(x), y)) & \\ \equiv & \exists w \neg P(a, w) \lor \forall x \exists y Q(f(x), y) & \text{(Scope)} \\ \equiv & \exists w \forall x \exists y(\neg P(a, w) \lor Q(f(x), y)) & \text{(Scope)} \\ \equiv_s & \forall x(\neg P(a, b) \lor Q(f(x), g(x))) & \text{(Skolemize: } \exists w \to b,\ \forall x \exists y \to g(x)\text{)} \end{array}$$

$$\begin{array}{lll} (b) & \neg Q(z) \lor \neg \exists x R(x, y) \lor \forall x \exists y P(x, g(y, f(x))) & \\ \equiv & \neg Q(z) \lor \forall x \neg R(x, y) \lor \forall x \exists y P(x, g(y, f(x))) & (\neg \exists x F \equiv \forall x \neg F) \\ \equiv & \neg Q(z) \lor \forall x \neg R(x, y) \lor \forall u \exists v P(u, g(v, f(u))) & \text{(Renaming of bound variables: } x \to u,\ y \to v\text{)} \\ \equiv_s & \exists z \neg Q(z) \lor \exists y \forall x \neg R(x, y) \lor \forall u \exists v P(u, g(v, f(u))) & \text{(Binding of free variables)} \\ \equiv & \forall u \exists v \exists y \forall x \exists z(Q(z) \lor \neg R(x, y) \lor P(u, g(v, f(u)))) & \text{(Scope)} \\ \equiv_s & \forall u \forall x(\neg Q(j(u, x)) \lor \neg R(x, i(u)) \lor P(u, g(h(u), f(u)))) & \text{(Skolemize: } v \to h(u),\ y \to i(u),\ z \to j(u, x)\text{)} \end{array}$$

Exercise 4 **3P**

Use resolution with unification to derive the empty clause from the following first-order formula F in clause normal form:

$$\{\ \{\neg P(f(y_1)), Q(y_1, h(z_1, z_1))\},\ \{\neg P(f(f(x_2))), \neg Q(f(x_2), y_2)\},$$
$$\{P(f(x_3)), Q(x_3, h(y_3, a))\},\ \{\neg Q(f(y_4), z_4), \neg Q(f(a), h(f(a), y_5))\}\ \}$$

In each step, clearly state (i) which variables are renamed before the computation of a most general unificator, (ii) which literals are unified, and (iii) which most general unificator is used for the resolution step.

Solution:

(a) $K_1 = \{\neg P(f(y_1)), Q(y_1, h(z_1, z_1))\}$, $K_2 = \{\neg P(f(f(x_2))), \neg Q(f(x_2), y_2)\}$.

As the set of variables occurring in K_1 is disjoint from that of K_2, we do not need to rename variables.

Unifying $\{Q(y_1, h(z_1, z_1)), Q(f(x_2), y_2)\}$ from left to right yields:

$[y_1/f(x_2)], [y_2/h(z_1, z_1)]$.

Resolvent: $K_5 = \{\neg P(f(f(x_2)))\}$.

(b) $K_3 = \{P(f(x_3)), Q(x_3, h(y_3, a))\}$, $K_4 = \{\neg Q(f(y_4), z_4), \neg Q(f(a), h(f(a), y_5))\}$

Again, no need to rename variables as the two sets are disjoint.

Unifying $\{Q(x_3, h(y_3, a)), Q(f(y_4), z_4), Q(f(a), h(f(a), y_5))\}$ from left to right leads to:

$[x_3/f(y_4)], [y_4/a], [z_4/h(y_3, a)], [y_3/f(a)], [y_5/a]$

Resolvent: $K_6 = \{P(f(f(a)))\}$.

(c) $K_5 = \{\neg P(f(f(x_2)))\}$, $K_6 = \{P(f(f(a)))\}$.

Again, no need to rename any variables.

Unifying $\{P(f(f(x_2))), P(f(f(a)))\}$ leads obviously to $[x_2/a]$.

Resolvent: $\square$.

Exercise 5 — 2P+3P+3P=8P

Syllogisms have been introduced at the beginning of the lecture as an example of logical inference. In terms of first order logic, a syllogism consists of three formulas F_1, F_2, F_3 – two premises F_1, F_2, and a conclusion F_3 – where each formula takes the form of one of the following formulas up to renaming the predicate symbols P, Q:

$$(1)\quad \forall x(P(x) \to Q(x)) \quad (2)\quad \forall x(P(x) \to \neg Q(x)) \quad (3)\quad \exists x(P(x) \wedge Q(x)) \quad (4)\quad \exists x(P(x) \wedge \neg Q(x)).$$

A syllogism F_1, F_2, F_3 is valid if $\models (F_1 \wedge F_2) \to F_3$; otherwise the syllogism is not valid.

Example: In case of the syllogism "If all men are mortal, and Socrates is a man, then Socrates is mortal" we have

$$F_1 = \forall x(\text{man}(x) \to \text{mortal}(x)),\ F_2 = \exists x(\text{Socrates}(x) \wedge \text{man}(x))),\quad \text{and } F_3 = \exists x(\text{Socrates}(x) \wedge \text{mortal}(x))$$

In this example, F_1 is of the form (1), while F_2, F_3 are both of form (3).

(a) Give an example of a syllogism which is not valid. Prove your answer correct.

(b) Give an example of a syllogism which is valid where (i) $F_1 \wedge F_2$ has to be satisfiable, (iii) F_1, F_2, F_3 have to be pairwise distinct formulas, and (iii) at least one formula of F_1, F_2, F_3 has to be of form (4).

Prove the correctness of your answer using resolution.

(c) Describe an algorithm that, on input a syllogism $(F_1 \wedge F_2) \to F_3$, always terminates and correctly outputs whether the syllogism is valid or not; if it is not valid, your algorithm should also output a suitable structure $\mathcal{A}$ with $\mathcal{A} \not\models (F_1 \wedge F_2) \to F_3$.

Hint: Recall that Gilmore's algorithm terminates if the Herbrand universe is finite.

Solution:

(a) Let $F_1 = F_2 = \exists x(A(x) \wedge \neg B(x)), F_3 = \forall x(A(x) \to B(x))$, $U_{\mathcal{A}} = \{a\}$, $A^{\mathcal{A}} = \{a\}$, $B^{\mathcal{A}} = \emptyset$.

Then $\mathcal{A} \models F_1 \wedge F_2$ but $\mathcal{A} \not\models (F_1 \wedge F_2) \to F_3$.

So, the syllogism is not valid.

(b) Let $F_1 = \forall(A(x) \to B(x))$, $F_2 = \exists y(A(y) \wedge \neg C(y))$, and $F_3 = \exists z(B(z) \wedge \neg C(z))$.

Then the so defined syllogism is valid iff $G = F_1 \wedge F_2 \wedge \neg F_3$ is unsatisfiable. We have

$$\begin{aligned} G &\equiv \exists y \forall x \forall z((\neg A(x) \vee B(x)) \wedge A(y) \wedge \neg C(y) \wedge (\neg B(z) \vee C(z))) \\ &\equiv_s \forall x \forall z((\neg A(x) \vee B(x)) \wedge A(a) \wedge \neg C(a) \wedge (\neg B(z) \vee C(z))) \\ &\equiv \{\ \{\neg A(x), B(x)\},\ \{A(a)\},\ \{\neg C(a)\},\ \{\neg B(z), C(z)\}\ \} \end{aligned}$$

Ground resolution yields:

$\mathsf{Res}(\{\neg A(x), B(x)\}, \{A(a)\}) = \{B(a)\}$

$\mathsf{Res}(\{B(a)\}, \{\neg B(z), C(z)\}) = \{C(a)\}$.

$\mathsf{Res}(\{C(a)\}, \{\neg C(a)\}) = \square$.

(c) Every formula F_i has no free variables. We therefore may assume that all formulas have disjoint sets of variables. Hence, we can always find for $(G = F_1 \wedge F_2) \rightarrow F_3$ an equivalent formula H in RPF with $H = \exists^*\forall^* H^*$. Hence, Skolemizing H to a formula S does not introduce a function symbol. Thus, $D(S)$ is finite and Gilmore's algorithm terminates and correctly decides whether H is unsatisfiable or not, i.e. whether the syllogism is valid or not. If H is satisfiable (i.e. the syllogism is not valid), we can simply enumerate the finite number of suitable Herbrand structures for H until we find a model for H, i.e. an counterexample for the syllogism.

Exercise 6 **4P**

For a propositional variable A and a propositional formula F, let $F[A/b]$ denote the propositional formula obtained from F by substituting the boolean value b for each occurrence of A in F – if A does not occur in F at all then $F[A/b] = F$.

Let F, H be propositional formulas. Assume $\models (H[A/0] \leftrightarrow H[A/1])$ and $\models H \rightarrow F$.

Show that also $\models H \rightarrow (F[A/0] \leftrightarrow F[A/1])$.

Remark: Let $\mathcal{A}$ be an assignment, A a propositional variable, and $b \in \{0,1\}$. Recall that $\mathcal{A}_{[A/b]}$ is the assignment with $\mathcal{A}_{[A/b]}(A) = b$ and $\mathcal{A}_{[A/b]}(B) = \mathcal{A}(B)$ for any propositional variable B distinct from A.

Start from a satisfying assignment $\mathcal{A}$ of H (i.e. $\mathcal{A}(H) = 1$), and use that $\mathcal{A}_{[A/b]}(G) = \mathcal{A}(G[A/b])$ for every $b \in \{0,1\}$ and every propositional formula G.

Solution: Let $\mathcal{A} \models H$ otherwise trivially $\mathcal{A} \models H \rightarrow (F[A/0] \leftrightarrow F[A/1])$.

Let $a := \mathcal{A}(A)$ and $\overline{a} := 1 - a$.

As $\models (H[A/0] \leftrightarrow H[A/1])$ we have $\mathcal{A}(H[A/a]) = \mathcal{A}(H[A/\overline{a}])$.

Note that $\mathcal{A}(H[A/b]) = \mathcal{A}_{[A/b]}(H)$ for $b \in \{0,1\}$.

Hence:

$1 = \mathcal{A}(H) = \mathcal{A}_{[A/a]}(H) = \mathcal{A}(H[A/a]) = \mathcal{A}(H[A/\overline{a}]) = \mathcal{A}_{[A/\overline{a}]}(H)$.

As $\models H \rightarrow F$ we have both $1 = \mathcal{A}_{[A/a]}(F) = \mathcal{A}(F[A/a])$ and $1 = \mathcal{A}_{[A/\overline{a}]}(F) = \mathcal{A}(F[A/\overline{a}])$.

So, $\mathcal{A}(F[A/a] \leftrightarrow F[A/\overline{a}]) = 1$.

Exercise 7 **2P+3P=5P**

(a) Let F be a first-order forumla in RPF, and G the formula obtained by Skolemizing F. It was shown in the lecture that any model $\mathcal{A}$ of G is also a model of F.

Show this result explicitly for the special case of $F = \forall x \exists y P(x,y)$ and $G = \forall x P(x, f(x))$.

(b) Show that any satisfiable formula F has an infinite model.

Hint: When exactly does the Herbrand universe $D(G)$ consist of infinitely many elements? For the case that $D(G)$ is finite, recall that, if $\mathcal{A} \models G \wedge H$, then also $\mathcal{A} \models G$.

Solution:

(a) Let $\mathcal{A} \models G$.

Then for every $d \in U_{\mathcal{A}}$ we have $(d, f^{\mathcal{A}}(d)) \in P^{\mathcal{A}}$,

i.e. for all $d \in U_{\mathcal{A}}$ we have $\mathcal{A}_{[x:=d, y:=f^{\mathcal{A}}(d)]} \models P(x,y)$,

i.e. for all $d \in U_{\mathcal{A}}$ we have $\mathcal{A}_{[x:=d]} \models \exists y P(x,y)$,

i.e. $\mathcal{A} \models \forall x \exists y P(x,y)$.

(b) As for every formula of first-order logic we can construct an equivalent one in RPF, we may assume that F is already in RPF.

Let G be the Skolemization of F. If G contains a function symbol, then $D(G)$ is infinite. As F is satisfiable, so is G; hence, there exists a Herbrand model for G and, thus, for F with an infinite universe.

Thus, assume G does not contain a function symbol. Let $G = \forall x_1 \dots \forall x_n G^*$, and P, f symbols not occurring in G.

Consider then $H = \forall y \forall x_1 \dots \forall x_n (G^* \wedge P(f(y))) \equiv G \wedge \forall y P(f(y))$.

As P, f do not occur in F, G, the formula H is still satisfiable: simply extend any model $\mathcal{A}$ of G to a model of H by $P^{\mathcal{A}} = U_{\mathcal{A}}$ and $f^{\mathcal{A}}$ the identity over $U_{\mathcal{A}}$.

As H is satisfiable and contains a function symbol, it has an infinite model, which is also a model of G and, thus, also a model of F.