Technische Universität München (I7) **Summer term 2011**
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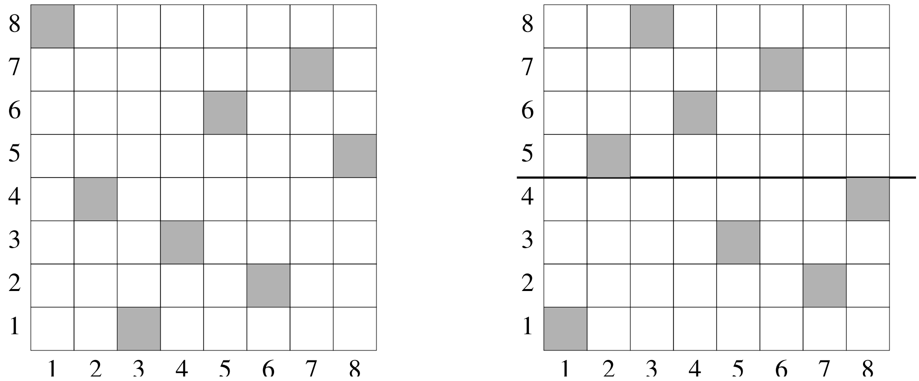
Logic – Homework 10

Discussed on .

Exercise 10.1 Eight Queens Problem 5P+2P=7P

The queen as a chess figure is a allowed to move arbitrary long moves in either vertical, horizontal or diagonal direction. The *Eight Queens Problem* then is as follows: On a normal chess-board with 8×8 fields, one wants to place eight queens in such a way, that it is not possible for any of these queens to attack another.

Below we present two different solutions:



(a) Create a propositional formula F that expresses the following statements:

i) $F_1 \hat{=}$ "in each row there is at least one queen"

ii) $F_2 \hat{=}$ "in each row there is at most one queen"

iii) $F_3 \hat{=}$ "in each column there is at most one queen"

iv) $F_4 \hat{=}$ "in each diagonal from top-left to bottom-right (NW-diagonal), there is at most one queen"

v) $F_5 \hat{=}$ "in each diagonal from bottom-left to top-right (NE-diagonal), there is at most one queen"

Use the variables x_{ij}, $1 \leq i, j \leq 8$ to state, that there is a queen at row i and col j.

Together these statements form the formula $F := F_1 \wedge F_2 \wedge F_3 \wedge F_4 \wedge F_5$, which describes all possible solutions, i.e. an assignment to F is a model iff the variables set to 1 are a solution to the eight queens problem.

Note: Two fields (i, j) and (i', j') are contained in the same NW-diagonal, iff $i + j = i' + j'$. Similarly, they are contained in the same NE-diagonal, iff $i - j = i' - j'$.

(b) The two boards presented above correlate via a *horizontal axis-symmetry.* This means, that if the one board is reflected along a horizontal axis through the center of the board (as sketched in the right picture), one receives the other one. It can be seen easily, that one board is a solution iff its mirrored counterpart is a solution.

Describe how the formula F needs to be altered, such that if two solutions are correlated via horizontal symmetry, then only one of them is a model of F.

Exercise 10.2 BDDs 2P+2P+3P=7P

(a) Recall the definition of the *if-then-else* operator ite:

$$\mathsf{ite}(F, G, H) \equiv (F \wedge G) \vee (\neg F \wedge H).$$

Show how to express $F \rightarrow G$ using only ite, F, G, and the constants 0 and 1 (representing false and true, respectively).

(b) W.r.t. the variable order $v < w < x < y < z$ construct the BDDs representing these two formulas

$$F_1 = \neg z \vee (v \wedge w) \text{ and } F_2 = (x \vee \neg z) \wedge (\neg x \vee \neg y).$$

(c) Construct the BDD for the formula $F = F_1 \vee F_2$. How many different assignments exist for F?

Exercise 10.3 **DPLL** **3P+2P=5P**

(a) Apply the DPLL-algorithm on the following formula F, that is give a maximal derivation for F.

Is F satisfiable? If yes, give a satisfying assignment.

$$F = \{\{\neg A, D\}, \{A, \neg B\}, \{\neg A, \neg D, \neg B\}, \{B, C\}, \{\neg A, B, \neg C, \neg D\}, \{A, D\}\}$$

(b) Recall the subsumption rule: *If a formula F contains two clauses C, C' with $C \subseteq C'$, then remove C' from F.*

Find a formula F that has the property, that there exists a derivation from F where the subsumption rule can be used, but there does not exist a derivation where it is used in the first step.

Exercise 10.4 **Unsatisfiability** **4P+4P=8P**

Let F be a propositional formula, which contains a variable A, and let $G := F[A/0] \wedge F[A/1]$, where $F[A/b]$ describes the formula, where every occurrence of A is replaced by b.

(a) Prove that $G \wedge \neg F$ is unsatisfiable.

(b) Let H be another formula, that does **not** contain the variable A. Then assume, that $H \wedge \neg F$ is unsatisfiable. Show that this implies, that $H \wedge \neg G$ is unsatisfiable.

Notes: Show in (a), that for each assignment $\mathcal{A}$ it holds that $\mathcal{A}(G \wedge \neg F) = 0$ by doing a case-destinction for $\mathcal{A}(A) = 0$ and $\mathcal{A}(A) = 1$. In (b) you can use (without proof), that for each formula F' it holds that, F' is unsatisfiable iff both $F'[A/0]$ and $F'[A/1]$ are unsatisfiable.

Exercise 10.5 **Predicate Logic** **2P+4P=6P**

(a) The following two formulas are given:

i) $F_1 = \forall x\big(P(x) \vee R(x)\big) \rightarrow \big(\forall x P(x) \wedge \forall x R(x)\big)$

ii) $F_2 = \forall x\big(P(x) \rightarrow Q(x)\big) \rightarrow \exists y\big(Q(y) \rightarrow P(y)\big)$

For each of these formulas state (if possible) a structure that satisfies the formula and one that does not.

(b) Let $F = \neg\exists x\big(P(x) \rightarrow \forall y P(y)\big)$.

Conduct the following tasks on F:

i) Transform F into a formula G in Skolem form **such that in G only nullary function symbols occur**.

ii) Enumerate all Herbrand structures of G and decide for each of them whether it is a model of G or not.

iii) State if by the results of (b) it follows that F is valid/satisfiable/unsatisfiable.

Exercise 10.6 **Resolution** **2P+3P+2P=7P**

Before a match of the national team of Germany, Jogi Löw announces the tactics and the current atmosphere in the team:

- Each forward (German: *Stürmer*) is in the starting lineup.
- No player in the starting lineup dislikes any other player in the starting lineup.
- Each player dislikes someone from the team.

A journalist concludes that each forward dislikes some non-forward. Is this correct?

(a) Formalize the statements of Jogi Löw as a formula F in predicate logic and the statement of the journalist as a formula J. Use the following predicates:

$$Fw(x)\text{: } x \text{ is a forward} \quad St(x)\text{: } x \text{ is in the starting lineup} \quad Dl(x,y)\text{: } x \text{ dislikes } y$$

(b) Transform the formula $F \wedge \neg J$ into an equisatisfiable (i.e. only equivalent up to satisfiability) formula H in Skolem form. State in each step if it results in a semantically equivalent or only in an equisatisfiable formula.

(c) Use resolution on H to derive the empty clause. What does this derivation of the empty clause imply for the conclusion of the journalist?