Technische Universität München (I7) **Summer term 2011**
Prof. Dr. Dr. h.c J. Esparza / M. Luttenberger / R. Neumann **Monday 1st August, 2011**

Solution

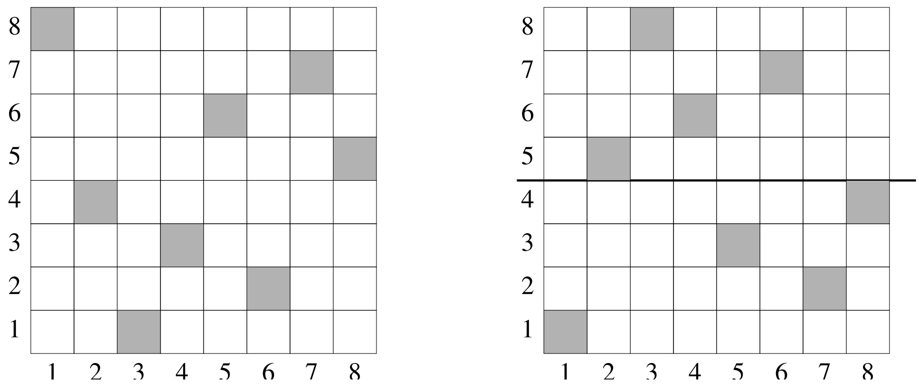
Logic – Homework 10

Discussed on .

Exercise 10.1 **Eight Queens Problem** **5P+2P=7P**

The queen as a chess figure is a allowed to move arbitrary long moves in either vertical, horizontal or diagonal direction. The *Eight Queens Problem* then is as follows: On a normal chess-board with 8×8 fields, one wants to place eight queens in such a way, that it is not possible for any of these queens to attack another.

Below we present two different solutions:



(a) Create a propositional formula F that expresses the following statements:

i) $F_1 \mathrel{\widehat{=}}$ "in each row there is at least one queen"

ii) $F_2 \mathrel{\widehat{=}}$ "in each row there is at most one queen"

iii) $F_3 \mathrel{\widehat{=}}$ "in each column there is at most one queen"

iv) $F_4 \mathrel{\widehat{=}}$ "in each diagonal from top-left to bottom-right (NW-diagonal), there is at most one queen"

v) $F_5 \mathrel{\widehat{=}}$ "in each diagonal from bottom-left to top-right (NE-diagonal), there is at most one queen"

Use the variables x_{ij}, $1 \leq i, j \leq 8$ to state, that there is a queen at row i and col j.

Together these statements form the formula $F := F_1 \wedge F_2 \wedge F_3 \wedge F_4 \wedge F_5$, which describes all possible solutions, i.e. an assignment to F is a model iff the variables set to 1 are a solution to the eight queens problem.

Note: Two fields (i, j) and (i', j') are contained in the same NW-diagonal, iff $i + j = i' + j'$. Similarly, they are contained in the same NE-diagonal, iff $i - j = i' - j'$.

(b) The two boards presented above correlate via a *horizontal axis-symmetry.* This means, that if the one board is reflected along a horizontal axis through the center of the board (as sketched in the right picture), one receives the other one. It can be seen easily, that one board is a solution iff its mirrored counterpart is a solution.

Describe how the formula F needs to be altered, such that if two solutions are correlated via horizontal symmetry, then only one of them is a model of F.

Solution:

(a) $$F_1 = \bigwedge_{i=1}^{8} \bigvee_{j=1}^{8} x_{ij}$$

$$F_2 = \bigwedge_{i=1}^{8} \bigwedge_{j=1}^{7} \bigwedge_{k=j+1}^{8} (x_{ij} \to \neg x_{ik})$$

$$F_3 = \bigwedge_{j=1}^{8} \bigwedge_{i=1}^{7} \bigwedge_{k=i+1}^{8} (x_{ij} \to \neg x_{kj})$$

$$F_4 = \bigwedge_{i=2}^{8} \bigwedge_{j=1}^{7} \bigwedge_{k=1}^{\min\{i-1,8-j\}} (x_{ij} \to \neg x_{i-k,j+k})$$

$$F_5 = \bigwedge_{i=1}^{7} \bigwedge_{j=1}^{7} \bigwedge_{k=1}^{\min\{8-i,8-j\}} (x_{ij} \to \neg x_{i+k,j+k})$$

In F_4 and F_5, we use the index k to iterate over the fields that are located to the right of (i, j); the minimum constraint guarantees, that the indices are in $[1, 8]$.

(b) In each column, there is exactly one queen. There are two cases:

- the queen is in the upper half, then it is placed in the lower half on the mirrored board
- the queen is in the lower half, then it is placed in the upper half on the mirrored board

Hence, we can demand, that in the lowest four fields of a row (let's use the first row), there is a queen:

$$F \wedge (x_{11} \vee x_{21} \vee x_{31} \vee x_{41})$$

Exercise 10.2 BDDs 2P+2P+3P=7P

(a) Recall the definition of the *if-then-else* operator ite:

$$\mathsf{ite}(F, G, H) \equiv (F \wedge G) \vee (\neg F \wedge H).$$

Show how to express $F \to G$ using only ite, F, G, and the constants 0 and 1 (representing false and true, respectively).

(b) W.r.t. the variable order $v < w < x < y < z$ construct the BDDs representing these two formulas

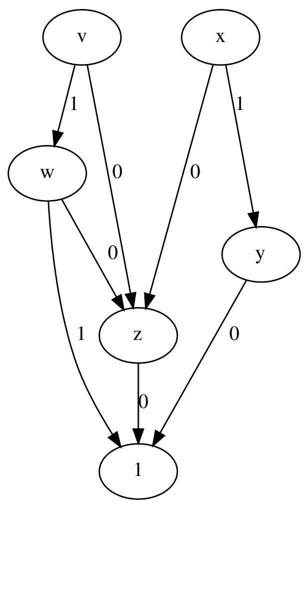
$$F_1 = \neg z \vee (v \wedge w) \text{ and } F_2 = (x \vee \neg z) \wedge (\neg x \vee \neg y).$$

(c) Construct the BDD for the formula $F = F_1 \vee F_2$. How many different assignments exist for F?

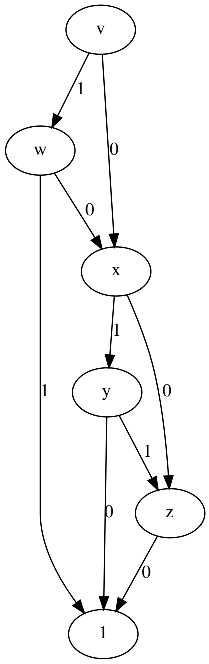
Solution:

(a) $\mathsf{ite}(F, G, 1)$

(b) Both BDDs presented in a multi-BDDs (the 0-node as been omitted):



(c) The BDD for $F_1 \vee F_2$:



Counting the satisfying assignments:

- Node z has a weight of 1.
- Node y has a weight of $2 \cdot 1 + 1 = 3$,
- Node x has a weight of $3 + 2 \cdot 1 = 5$,
- Node w has a weight of $5 + 8 \cdot 1 = 13$,
- Node v has a weight of $2 \cdot 5 + 13 = 23$.

Hence there are 23 satisfying assignments.

Exercise 10.3 DPLL 3P+2P=5P

(a) Apply the DPLL-algorithm on the following formula F, that is give a maximal derivation for F.

Is F satisfiable? If yes, give a satisfying assignment.

$$F = \big\{\{\neg A, D\}, \{A, \neg B\}, \{\neg A, \neg D, \neg B\}, \{B, C\}, \{\neg A, B, \neg C, \neg D\}, \{A, D\}\big\}$$

(b) Recall the subsumption rule: *If a formula F contains two clauses C, C' with $C \subseteq C'$, then remove C' from F.*

Find a formula F that has the property, that there exists a derivation from F where the subsumption rule can be used, but there does not exist a derivation where it is used in the first step.

Solution:

(a) We start with the block $\{F\}$, which unfolded is

$$\Big\{\{\{\neg A, D\},\ \{A, \neg B\},\ \{\neg A, \neg D, \neg B\},\ \{B, C\},\ \{\neg A, B, \neg C, \neg D\},\ \{A, D\}\}\Big\}$$

Applying the splitting rule on A and thereafter applying the one-literal-rule, we receive a block with two formulas, namely on for the case where we assume that A is set to true, and one for the case where $\neg A$ is assumed to be true:

$$\Big\{\{\{D\},\ \{\neg D, \neg B\},\ \{B, C\},\ \{B, \neg C, \neg D\}\},\ \ \{\{\neg B\},\ \{B, C\},\ \{D\}\}\Big\}$$

On both formulas we can apply the single-literal-rule using D:

$$\Big\{\{\{\neg B\},\ \{B,C\},\ \{B,\neg C\}\},\ \ \{\{\neg B\},\ \{B,C\}\}\Big\}$$

Again the single-literal-rule, this time using $\neg B$:

$$\Big\{\{\{C\},\ \{\neg C\}\},\ \ \{\{C\}\}\Big\}$$

And again with C:

$$\Big\{\{\emptyset\},\ \ \emptyset\Big\}$$

It is not possible to apply further rules, therefore the derivation is maximal. It is also satisfying as the last block contains the empty formula. Hence the formula is satisfiable. A satisfying assignment is $A = 0, B = 0, C = 1, D = 1$, which can be deduced from the steps needed to reach the empty formula.

(b) Let $F = \big\{\{\neg A\},\ \{A,C\},\ \{B,C\}\big\}$.

No clause is a subset of any other one, hence the subsumption rule cannot be applied. After one uses the single-literal-clause using $\neg A$, one gets a block with the formula $\big\{\{C\},\ \{B,C\}\big\}$. And on this block the subsumption rule can finally be applied.

Exercise 10.4 Unsatisfiability **4P+4P=8P**

Let F be a propositional formula, which contains a variable A, and let $G := F[A/0] \wedge F[A/1]$, where $F[A/b]$ describes the formula, where every occurrence of A is replaced by b.

(a) Prove that $G \wedge \neg F$ is unsatisfiable.

(b) Let H be another formula, that does **not** contain the variable A. Then assume, that $H \wedge \neg F$ is unsatisfiable. Show that this implies, that $H \wedge \neg G$ is unsatisfiable.

Notes: Show in (a), that for each assignment $\mathcal{A}$ it holds that $\mathcal{A}(G \wedge \neg F) = 0$ by doing a case-destinction for $\mathcal{A}(A) = 0$ and $\mathcal{A}(A) = 1$. In (b) you can use (without proof), that for each formula F' it holds that, F' is unsatisfiable iff both $F'[A/0]$ and $F'[A/1]$ are unsatisfiable.

Solution:

(a) Let $\mathcal{A}$ be an arbitrary assignment suitable for $G \wedge \neg F$. Let $b := \mathcal{A}(A)$ and write $G \wedge \neg F \equiv F[A/1-b] \wedge (F[A/b] \wedge \neg F)$. We further have $\mathcal{A}(F) = \mathcal{A}(F[A/b])$, and thus $\mathcal{A}(\neg F) = 1 - \mathcal{A}(F[A/b])$, i.e., $\mathcal{A}(F[A/b] \wedge \neg F) = 0$.

(b) It holds:

$$\begin{aligned} H \wedge \neg G &\equiv H \wedge \neg(F[A/0] \wedge F[A/1]) \\ &\equiv H \wedge (\neg F[A/0] \vee \neg F[A/1]) \\ &\equiv (H \wedge \neg F[A/0]) \vee (H \wedge \neg F[A/1]) \end{aligned}$$

As A does not occur in H, the last line is equivalent to:

$$(H \wedge \neg F)[A/0] \vee (H \wedge \neg F)[A/1] =: J$$

If two formulas are each unsatisfiable, so is their disjunction. Therefore it follows, that if $H \wedge \neg F$ is unsatisfiable, so is J.

Alternatively:

We show that $\models H \to G$ under the stated assumptions that A does not occur in H, and $\models H \to F$.

Let $\mathcal{A}$ be any assignment defined on the variables occurring in $H \to G$.

If $\mathcal{A}(H) = 0$, then trivially $\mathcal{A} \models H \to G$ and we are done. So assume $\mathcal{A}(H) = 1$.

At least A does not occur in $H \to G$, so extend $\mathcal{A}$ to an assignment $\mathcal{B}$ suitable also for F by choosing *arbitrary* values for those variables occurring only in F so that $\mathcal{A}$ and $\mathcal{B}$ coincide on the variables of $H \to G$ and $\mathcal{A}(H) = \mathcal{B}(H)$ and $\mathcal{A}(G) = \mathcal{B}(G)$.

As (i) $\mathcal{B}(H) = 1$, (ii) $\mathcal{B}$ is suitable for $H \to F$, (iii) $\models H \to F$, and (iv) $\mathcal{B}(A)$ was chosen arbitrarily, we conclude $\mathcal{B}(F) = 1$ *independently* of the choice of $\mathcal{B}(A)$.

Hence, $\mathcal{B}(F[A/1]) = \mathcal{B}(F[A/0]) = 1$ and $\mathcal{B}(G) = 1$. As $\mathcal{A}$ and $\mathcal{B}$ coincide on the variables occurring in G, also $\mathcal{A}(G) = 1$ so that $\mathcal{A} \models H \to G$.

Exercise 10.5 **Predicate Logic** **2P+4P=6P**

(a) The following two formulas are given:

i) $F_1 = \forall x\big(P(x) \lor R(x)\big) \to \big(\forall x P(x) \land \forall x R(x)\big)$

ii) $F_2 = \forall x\big(P(x) \to Q(x)\big) \to \exists y\big(Q(y) \to P(y)\big)$

For each of these formulas state (if possible) a structure that satisfies the formula and one that does not.

(b) Let $F = \neg\exists x\big(P(x) \to \forall y P(y)\big)$.

Conduct the following tasks on F:

i) Transform F into a formula G in Skolem form **such that in G only nullary function symbols occur**.

ii) Enumerate all Herbrand structures of G and decide for each of them whether it is a model of G or not.

iii) State if by the results of (b) it follows that F is valid/satisfiable/unsatisfiable.

Solution:

(a) Model of F_1: $U_{\mathcal{A}} = \{1\},\ P^{\mathcal{A}} = R^{\mathcal{A}} = \emptyset$
Model of $\neg F_1$: $U_{\mathcal{B}} = \{1\},\ P^{\mathcal{B}} = \{1\},\ R^{\mathcal{B}} = \emptyset$
Model of F_2: $U_{\mathcal{C}} = \{1\},\ P^{\mathcal{C}} = Q^{\mathcal{C}} = \emptyset$
Model of $\neg F_2$: $U_{\mathcal{D}} = \{1\},\ P^{\mathcal{D}} = \emptyset,\ Q^{\mathcal{D}} = \{1\}$

(b) i) We will exploit the fact, that x does not occur freely in $\exists y \neg P(y)$ and neither does y in $\forall x P(x)$:

$$\begin{aligned} F &\equiv \forall x\big(P(x) \land \exists y \neg P(y)\big) \\ &\equiv \forall x P(x) \land \exists y \neg P(y) \\ &\equiv \exists y \forall x\big(P(x) \land \neg P(y)\big) \\ &\equiv_S \forall x\big(P(x) \land \neg P(a)\big) =: G \end{aligned}$$

ii) The Herbrand universe of G is $D(G) = \{a\}$. Therefore G has two Herbrand structures $\mathcal{A}$ and $\mathcal{B}$ where $U_{\mathcal{A}} = U_{\mathcal{B}} = D(G) = \{a\}$, and $P^{\mathcal{A}} = \emptyset$ and $P^{\mathcal{B}} = \{a\}$, respectively.

iii) Both $\mathcal{A}$ and $\mathcal{B}$ are not models of G. From the fundamental theorem of predicate logic it follows that G is unsatisfiable. As $G \equiv_S F$, F is also unsatisfiable.

Exercise 10.6 **Resolution** **2P+3P+2P=7P**

Before a match of the national team of Germany, Jogi Löw announces the tactics and the current atmosphere in the team:

- Each forward (German: *Stürmer*) is in the starting lineup.
- No player in the starting lineup dislikes any other player in the starting lineup.
- Each player dislikes someone from the team.

A journalist concludes that each forward dislikes some non-forward. Is this correct?

(a) Formalize the statements of Jogi Löw as a formula F in predicate logic and the statement of the journalist as a formula J. Use the following predicates:

$Fw(x)$: x is a forward $St(x)$: x is in the starting lineup $Dl(x, y)$: x dislikes y

(b) Transform the formula $F \land \neg J$ into an equisatisfiable (i.e. only equivalent up to satisfiability) formula H in Skolem form. State in each step if it results in a semantically equivalent or only in an equisatisfiable formula.

(c) Use resolution on H to derive the empty clause. What does this derivation of the empty clause imply for the conclusion of the journalist?

Solution:

(a) The statements can be formalized as follows:

$$\begin{aligned}F &= \forall x\big(Fw(x) \to St(x)\big)\\ &\quad \wedge \forall x \forall y\big((St(x) \wedge St(y)) \to \neg Ds(x,y)\big)\\ &\quad \wedge \forall x \exists y Ds(x,y)\\ J &= \forall x\big(Fw(x) \to \exists y(\neg Fw(y) \wedge Ds(x,y))\big)\end{aligned}$$

Remark: $J_2 = \forall x \exists y((Fw(x) \wedge \neg Fw(y)) \to Ds(x,y))$ does not correctly model the statement of the journalist as it is in fact a tautology: Let $\mathcal{A}$ be any suitable structure for J_2 and choose any $d \in U_{\mathcal{A}}$. We need to show that we can find an $e_d \in U_{\mathcal{A}}$ such that $\mathcal{A}_{[x/d][y/e_d]} \models (Fw(x) \wedge \neg Fw(y)) \to Ds(x,y)$.

If $Fw^{\mathcal{A}} = \emptyset$, then we can choose any e as $\mathcal{A}_{[x/d]} \not\models Fw(x)$; otherwise, we can simply choose $e_d \in Fw^{\mathcal{A}}$ so that $\mathcal{A}_{[x/d][y/e_d]} \not\models \neg Fw(y)$. In both cases, $\mathcal{A}_{[x/d][y/e_d]} \not\models Fw(x) \wedge \neg Fw(y)$ and thus trivially $\mathcal{A}_{[x/d][y/e]} \models (Fw(x) \wedge \neg Fw(y)) \to Ds(x,y)$.

Note that this also nicely illustrates why you need to move to $\neg J_2$ in order to use the fundamental theorem for showing that J_2 is valid; skolemizing J_2 would introduce a function symbol which restricts us from freely choosing the e_d as done above.

$$\neg J_2 \equiv \exists x \forall y(Fw(x) \wedge \neg Fw(y) \wedge \neg Ds(x,y)) \equiv_s \forall y(Fw(a) \wedge \neg Fw(y) \wedge \neg Ds(x,y)) =: SJ_2.$$

Now it follows analogously to above and Ex. 10.5 that SJ_2 is unsatisfiable as every Herbrand structure is not a model, and thus J_2 is valid.

(b)

$$\begin{aligned}F \wedge \neg J &\equiv \forall x\big(Fw(x) \to St(x)\big)\\ &\quad \wedge \forall x \forall y\big((St(x) \wedge St(y)) \to \neg Ds(x,y)\big)\\ &\quad \wedge \forall x \exists u G(x,u)\\ &\quad \wedge \exists v\big(Fw(v) \wedge \forall x(Fw(x) \vee \neg Ds(v,x))\big)\\ &\equiv \exists v \forall x \exists u \forall y\\ &\quad \Big(\big(\neg Fw(x) \vee St(x)\big)\\ &\quad \wedge \big(\neg St(x) \vee \neg St(y) \vee \neg Ds(x,y)\big)\\ &\quad \wedge Ds(x,u)\\ &\quad \wedge \big(Fw(v) \wedge (Fw(x) \vee \neg Ds(v,x))\big)\Big)\\ &\equiv_S \forall x \forall y\\ &\quad \Big(\big(\neg Fw(x) \vee St(x)\big)\\ &\quad \wedge \big(\neg St(x) \vee \neg St(y) \vee \neg Ds(x,y)\big)\\ &\quad \wedge Ds(x,f(x))\\ &\quad \wedge Fw(a)\\ &\quad \wedge \big(Fw(x) \vee \neg Ds(a,x)\big)\Big)\end{aligned}$$

(c) Name the clauses of above formula as follows:

$$\begin{aligned}C_1 &= \{\neg Fw(x), St(x)\}\\ C_2 &= \{\neg St(x), \neg St(y), \neg Ds(x,y)\}\\ C_3 &= \{Ds(x,f(x))\}\\ C_4 &= \{Fw(a)\}\\ C_5 &= \{\neg Ds(x,y), Fw(x)\}\end{aligned}$$

Linear resolution:

$$\begin{aligned}C_6 &= \{St(a)\} = (C_1 - \{\neg Fw(x)\})[][x/a] \cup (C_4 - \{Fw(a)\})[][x/a]\\ C_7 &= \{\neg St(y), \neg Ds(a,y)\} = (C_2 - \{\neg St(x)\})[][x/a] \cup (C_6 - \{St(a)\})[][x/a]\\ C_8 &= \{\neg St(f(a))\} = (C_3 - \{Ds(x,f(x))\})[][x/a][y/f(a)] \cup (C_7 - \{\neg Ds(a,y)\})[][x/a][y/f(a)]\\ C_9 &= \{\neg Fw(f(a))\} = (C_1 - \{St(x)\})[][x/f(a)] \cup (C_8 - \{\neg St(f(a))\})[][x/f(a)]\\ C_{10} &= \{\neg Ds(a,f(a))\} = (C_5 - \{Fw(x)\})[][x/f(a)] \cup (C_9 - \{\neg Fw(f(a))\})[][x/f(a)]\\ C_{11} &= \square = (C_3 - \{Ds(x,f(x))\})[][x/a] \cup (C_{10} - \{Ds(a,f(a))\})[][x/a]\end{aligned}$$