Technische Universität München **Summer term 2011**
Theoretische Informatik
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Logic – Endterm

***Please note*: If not stated otherwise, all answers have to be justified.**

Exercise 1 2P+4P=6P

(a) Recall the definition of the *if-then-else* operator ite:

$$\mathsf{ite}(F, G, H) \equiv (F \wedge G) \vee (\neg F \wedge H).$$

Show how to express $A \leftrightarrow \neg B$ using only ite, A, B, and the constants 0 and 1 (representing false and true, respectively).

Prove that your formula is equivalent to $A \leftrightarrow \neg B$ using equivalence transformations.

(b) W.r.t. the variable order $x < y < r < c$ construct the BDD representing the following formula:

$$F = (r \leftrightarrow (x \leftrightarrow \neg y)) \wedge (c \leftrightarrow (x \wedge y)).$$

Exercise 2 4P

Skolemize the following formula. In every step, state how the formula was transformed and whether semantic equivalence or only equisatisfiability holds:

$$F = \neg \exists x \forall y (P(x, y) \wedge \exists x (P(x, x) \rightarrow Q(z))).$$

Exercise 3 4P

Consider the following formulas where a, b are constants, and P, Eq are predicate symbols:

$$\begin{aligned} F_1 &= \forall x \forall y \forall z \forall v \ (\ (P(x, y, z) \wedge P(x, y, v)) \rightarrow Eq(z, v) \ ) \\ F_2 &= \forall x \ (\ P(x, a, x) \wedge P(b, x, x) \ ) \\ F_3 &= Eq(b, a) \end{aligned}$$

Show that $G = (F_1 \wedge F_2) \rightarrow F_3$ is valid using resolution.

Remark: State clearly intermediate results so that if you make a mistake you do not lose all points.

Exercise 4 2P+1P+3P=6P

(a) Assume F is a satisfiable formula of first-order logic *in clause form* with an infinite Herbrand universe.

Is it true that every model of F has an infinite universe? Prove your answer correct.

(b) Give an example of a satisfiable formula F (w/o equality!, not necessarily in clause form) such that every model of F has an infinite universe.

Remark: You do not have to prove that your formula has the required property.

(c) Skolemize the formula

$$F = \exists x P(x) \vee \exists y P(y) \vee \forall z P(z)$$

in three different ways yielding formulas G_1, G_2, G_3 such that for the Herbrand universe $D(G_i)$ it holds that

i) $D(G_1)$ consists of exactly one element,

ii) $D(G_2)$ consists of exactly two elements, and

iii) $D(G_3)$ is infinite.

Exercise 5 **2P+2P+2P=6P**

The semantics of the uniqueness quantifier $\exists! x$ (read: there exists a unique x such that ...) is defined as follows:

$$\mathcal{A} \models \exists! x F \quad \textbf{if and only if} \quad \text{there exists } d_0 \in U_{\mathcal{A}} \text{ such that } \mathcal{A}_{[x:=d_0]} \models F \text{ and for all } d \in U_{\mathcal{A}} \text{ if } \mathcal{A}_{[x:=d]} \models F, \text{ then } d = d_0.$$

Prove each of the nonequivalences stated below: That is, for each nonequivalence $Qx\exists! yF \not\equiv \exists! yQxF$ give a formula F and a structure $\mathcal{A}$ so that $\mathcal{A}$ is suitable for both formulas $Qx\exists! yF$ and $\exists! yQxF$, but only a model for one of them.

$$(a) \quad \forall x\exists! yF \not\equiv \exists! y\forall xF \qquad (b) \quad \exists x\exists! yF \not\equiv \exists! y\exists xF \qquad (c) \quad \exists! x\exists! yF \not\equiv \exists! y\exists! xF.$$

Remark: Try to interpret the formulas as statements on directed graphs.

Exercise 6 **7P**

For each of the following sets $\mathbf{L}$ of literals compute (from left to right, as in the algorithm discussed in the lecture) a most general unificator sub and the result $\mathbf{L}\mathsf{sub}$ of the unification **if** sub exists; otherwise state why sub does not exists.

(a) $\mathbf{L} = \{P(\ g(f(x_1), x_2),\ f(g(x_1, x_2))\ ), \quad P(\ g(y_1, f(y_2)),\ f(g(y_3, y_4))\ )\}$.

(b) $\mathbf{L} = \{P(\ g(x_1, f(x_2)),\ f(g(x_3, x_2))\ ), \quad P(\ g(y_1, f(y_2)),\ f(g(y_3, f(y_2)))\ )\}$.

(c) $\mathbf{L} = \{P(\ g(f(x_1), x_2),\ f(g(x_1, x_2))\ ), \quad P(\ g(y_1, y_3),\ f(y_5)\ ), \quad P(\ g(y_1, f(y_2)),\ f(g(y_3, y_4))\ )\}$.

(d) $\mathbf{L} = \{\neg P(\ g(f(x_1), x_2),\ f(g(x_1, x_2))\ ), \quad P(\ g(y_1, y_3),\ f(y_5)\ )\}$.

Exercise 7 **1P+1P+1P=3P**

Let $T_1 \subseteq T_2$ be two theories of first-order logic.

Prove or refute each of the following statements:

(a) If T_2 is decidable, then so is T_1.

(b) If T_2 is complete, then so is T_1.

(c) If T_2 is consistent, then so is T_1.

Remark: Only yes/no does not suffice, you have to explain why the statement holds or does not hold.

Exercise 8 **2P+2P=4P**

Given a set $\mathcal{X}$ of propositional formulas, let $\mathsf{Cn}(\mathcal{X})$ denote the set of consequences of $\mathcal{X}$, i.e., the set of all propositional formulas F with $\mathcal{X} \models F$.

Let $\mathcal{X}$ be an arbitrary set of propositional formulas, and $\mathcal{Y} = \{F_1, \ldots, F_n\}$ a finite set of propositional formulas (not necessarily included in $\mathcal{X}$) such that $\mathsf{Cn}(\mathcal{X}) = \mathsf{Cn}(\mathcal{Y})$.

(a) Prove that there is a **finite** subset $\mathcal{X}' \subseteq \mathcal{X}$ such that $\mathsf{Cn}(\mathcal{X}') = \mathsf{Cn}(\mathcal{Y})$ **using the compactness theorem**.

(b) Give an alternative proof for the result of (a) but this time **based on the results regarding the Hilbert calculus**.