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Theoretische Informatik
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SOLUTION

Logic – Endterm

***Please note*: If not stated otherwise, all answers have to be justified.**

Exercise 1 **2P+4P=6P**

(a) Recall the definition of the *if-then-else* operator ite:

$$\mathsf{ite}(F, G, H) \equiv (F \wedge G) \vee (\neg F \wedge H).$$

Show how to express $A \leftrightarrow \neg B$ using only ite, A, B, and the constants 0 and 1 (representing false and true, respectively).

Prove that your formula is equivalent to $A \leftrightarrow \neg B$ using equivalence transformations.

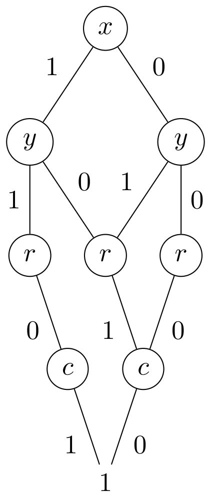
(b) W.r.t. the variable order $x < y < r < c$ construct the BDD representing the following formula:

$$F = (r \leftrightarrow (x \leftrightarrow \neg y)) \wedge (c \leftrightarrow (x \wedge y)).$$

Solution:

(a) $A \leftrightarrow \neg B \equiv (A \wedge \neg B) \vee (\neg A \wedge B) \equiv \mathsf{ite}(A, \neg B, B) \equiv \mathsf{ite}(A, \mathsf{ite}(B, 0, 1), B)$.

(b) Edges to the 0-node have been omitted:



Exercise 2 **4P**

Skolemize the following formula. In every step, state how the formula was transformed and whether semantic equivalence or only equisatisfiability holds:

$$F = \neg \exists x \forall y (P(x, y) \wedge \exists x (P(x, x) \rightarrow Q(z))).$$

Solution:

$$\begin{array}{rcl} F & = & \neg\exists x\forall y(P(x,y) \wedge \exists x(P(x,x) \rightarrow Q(z))) \\ & \equiv & \forall x\exists y(\neg P(x,y) \vee \forall x(P(x,x) \wedge \neg Q(z))) \\ & \equiv & \forall x\exists y(\neg P(x,y) \vee \forall u(P(u,u) \wedge \neg Q(z))) \\ & \equiv & \forall x\exists y\forall u(\neg P(x,y) \vee (P(u,u) \wedge \neg Q(z))) \\ & \equiv_s & \exists z\forall x\exists y\forall u(\neg P(x,y) \vee (P(u,u) \wedge \neg Q(z))) \\ & \equiv_s & \forall x\forall u(\neg P(x,f(x)) \vee (P(u,u) \wedge \neg Q(a))). \end{array}$$

Exercise 3 **4P**

Consider the following formulas where a, b are constants, and P, Eq are predicate symbols:

$$\begin{array}{rcl} F_1 & = & \forall x\forall y\forall z\forall v\ (\ (P(x,y,z) \wedge P(x,y,v)) \rightarrow Eq(z,v)\ ) \\ F_2 & = & \forall x\ (\ P(x,a,x) \wedge P(b,x,x)\ ) \\ F_3 & = & Eq(b,a) \end{array}$$

Show that $G = (F_1 \wedge F_2) \rightarrow F_3$ is valid using resolution.

Remark: State clearly intermediate results so that if you make a mistake you do not lose all points.

Solution: We prove that $\neg G$ is unsatisfiable. We have $\neg G \equiv F_1 \wedge F_2 \wedge \neg F_3$, and so the clauses of $\neg G$ are the union of the clauses of F_1, F_2, and $\neg F_3$. Further

$$F_1 \equiv \forall x\forall y\forall z\ (\ \neg P(x,y,z) \vee \neg P(x,y,v) \vee Eq(z,v)\ )$$

and so F_1 has only one clause

$$F_1 \equiv \quad \{\neg P(x,y,z), \neg P(x,y,v), Eq(z,v)\}$$

The clauses for F_2 are

$$F_2 \equiv \quad \{P(w,a,w)\}\ \ \{P(b,u,u)\}$$

The clause for $\neg F_3$ is

$$\neg F_3 \equiv \{\neg Eq(b,a)\}$$

Unifying $P(x,y,z)$ and $P(w,a,w)$ yields the clause

$$\{\neg P(w,a,v), Eq(w,v)\}$$

Unifying $P(b,u,u)$ and $P(w,r,v)$ yields the clause

$$\{Eq(b,a)\}$$

Resolving with the clause of $\neg F_3$ yields the empty clause.

Exercise 4 **2P+1P+3P=6P**

(a) Assume F is a satisfiable formula of first-order logic *in clause form* with an infinite Herbrand universe.

Is it true that every model of F has an infinite universe? Prove your answer correct.

(b) Give an example of a satisfiable formula F (w/o equality!, not necessarily in clause form) such that every model of F has an infinite universe.

Remark: You do not have to prove that your formula has the required property.

(c) Skolemize the formula

$$F = \exists x P(x) \vee \exists y P(y) \vee \forall z P(z)$$

in three different ways yielding formulas G_1, G_2, G_3 such that for the Herbrand universe $D(G_i)$ it holds that

i) $D(G_1)$ consists of exactly one element,

ii) $D(G_2)$ consists of exactly two elements, and

iii) $D(G_3)$ is infinite.

Solution:

(a) No, as $F = P(f(a))$ has as Herbrand universe $D(F) = \{a, f(a), f(f(a)), \ldots, f^k(a), \ldots\}$, but $U_\mathcal{A} = \{0\}$ with $f^\mathcal{A}(0) = 0$ and $P^\mathcal{A} = \{0\}$ is a finite model.

(b)

$$F = \forall x \forall y \forall z((\neg \mathsf{Lt}(x,y) \vee \neg \mathsf{Lt}(y,x)) \wedge (\neg \mathsf{Lt}(x,y) \vee \neg \mathsf{Lt}(y,z) \vee \mathsf{Lt}(x,z)) \wedge \exists w \mathsf{Lt}(x,w)).$$

(c) i)

$$\begin{aligned} F &\equiv \exists x P(x) \vee \exists x P(x) \vee \forall z P(z) && \text{renaming } y \text{ to } x \\ &\equiv \exists x P(x) \vee \forall z P(z) && \text{idempotence} \\ &\equiv \exists x \forall z (P(x) \vee P(z)) && \text{RPF} \\ &\equiv_s \forall z (P(a) \vee P(z)) && \text{skolemized} \\ &:= F_1 \end{aligned}$$

$$D(F_1) = \{a\}$$

ii)

$$\begin{aligned} F &\equiv \exists x \exists y \forall z (P(x) \vee P(y) \vee P(z)) && \text{RPF} \\ &\equiv_s \forall z (P(a) \vee P(b) \vee P(z)) && \text{skolemized} \\ &:= F_2 \end{aligned}$$

$$D(F_2) = \{a, b\}$$

iii)

$$\begin{aligned} F &\equiv \exists x P(x) \vee \exists x P(x) \vee \forall z P(z) && \text{renaming } y \text{ to } x \\ &\equiv \exists x P(x) \vee \forall z P(z) && \text{idempotence} \\ &\equiv \forall z \exists x (P(x) \vee P(z)) && \text{RPF} \\ &\equiv_s \forall z (P(f(z)) \vee P(z)) && \text{skolemized} \\ &:= F_3 \end{aligned}$$

$$D(F_3) = \{a, f(a), f(f(a)), \ldots\}$$

Note that the order of the quantifiers in the RPF is different from i).

Exercise 5 **2P+2P+2P=6P**

The semantics of the uniqueness quantifier $\exists! x$ (read: there exists a unique x such that ...) is defined as follows:

$\mathcal{A} \models \exists! x F$ **if and only if** there exists $d_0 \in U_\mathcal{A}$ such that $\mathcal{A}_{[x:=d_0]} \models F$ and for all $d \in U_\mathcal{A}$ if $\mathcal{A}_{[x:=d]} \models F$, then $d = d_0$.

Prove each of the nonequivalences stated below: That is, for each nonequivalence $Qx\exists! yF \not\equiv \exists! yQxF$ give a formula F and a structure $\mathcal{A}$ so that $\mathcal{A}$ is suitable for both formulas $Qx\exists! yF$ and $\exists! yQxF$, but only a model for one of them.

$(a)\ \forall x \exists! y F \not\equiv \exists! y \forall x F$ $(b)\ \exists x \exists! y F \not\equiv \exists! y \exists x F$ $(c)\ \exists! x \exists! y F \not\equiv \exists! y \exists! x F.$

Remark: Try to interpret the formulas as statements on directed graphs.

Solution: Let $F = P(x, y)$ and read P as the edge relation of a directed graph. The structure $\mathcal{A}$ given shall always be a model of the LHS but not of the RHS.

(a) LHS states that every node has a unique successor, while RHS states that there is a unique node which is a successor of every node:

$$U_\mathcal{A} = \{a, b\} \quad P^\mathcal{A} = \{(a, b), (b, a)\}$$

(b) LHS states that there is some node which has a unique successor, while RHS states that there is a unique node which has a predecessor.

$$U_{\mathcal{A}} = \{a, b\} \quad P^{\mathcal{A}} = \{(a, b), (b, a)\}$$

(c) LHS states that there is a unique node which has a unique successor, while RHS states that there is a unique node which has a unique predecessor.

$$U_{\mathcal{A}} = \{a, b, c\} \quad P^{\mathcal{A}} = \{(a, b), (b, b), (c, c), (b, c)\}$$

Exercise 6 **7P**

For each of the following sets **L** of literals compute (from left to right, as in the algorithm discussed in the lecture) a most general unificator sub and the result **L**sub of the unification **if** sub exists; otherwise state why sub does not exists.

(a) $\mathbf{L} = \{P(\ g(f(x_1), x_2),\ f(g(x_1, x_2))\ ),\quad P(\ g(y_1, f(y_2)),\ f(g(y_3, y_4))\ )\}$.

(b) $\mathbf{L} = \{P(\ g(x_1, f(x_2)),\ f(g(x_3, x_2))\ ),\quad P(\ g(y_1, f(y_2)),\ f(g(y_3, f(y_2)))\ )\}$.

(c) $\mathbf{L} = \{P(\ g(f(x_1), x_2),\ f(g(x_1, x_2))\ ),\quad P(\ g(y_1, y_3),\ f(y_5)\ ),\quad P(\ g(y_1, f(y_2)),\ f(g(y_3, y_4))\ )\}$.

(d) $\mathbf{L} = \{\neg P(\ g(f(x_1), x_2),\ f(g(x_1, x_2))\ ),\quad P(\ g(y_1, y_3),\ f(y_5)\ )\}$.

Solution:

(a) sub $= [y_1/f(x_1)][x_2/f(y_2)][y_3/x_1][y_4/f(y_2)]$ **L**sub $= \{P(\ g(f(x_1), f(y_2)),\ f(g(x_1, f(y_2)))\ )\}$

(b) $[x_1/y_1][x_2/y_2][x_3/y_3]$ – then you have to unify $f(y_2)$ with y_2, which is not possible, as y_2 occurs in $f(y_2)$

(c) sub $= [y_1/f(x_1)][y_3/f(y_2)][x_2/f(y_2)][y_5/g(x_1, f(y_2))][x_1/f(y_2)][y_4/f(y_2)]$
Lsub $= \{P(\ g(f(f(y_2)), f(y_2)),\ f(g(f(y_2), f(y_2)))\ )\}$

(d) not unifiable: you cannot unify P and $\neg P$

Exercise 7 **1P+1P+1P=3P**

Let $T_1 \subseteq T_2$ be two theories of first-order logic.

Prove or refute each of the following statements:

(a) If T_2 is decidable, then so is T_1.

(b) If T_2 is complete, then so is T_1.

(c) If T_2 is consistent, then so is T_1.

Remark: Only yes/no does not suffice, you have to explain why the statement holds or does not hold.

Solution:

(a) Let F be a formula of arithmetic. Then the theory with axioms $\{F, \neg F\}$ is decidable and a superset of arithmetic, while arithmetic is not decidable.

(b) Arithmetic is complete, but any subtheory induced by finitely many axioms is not.

(c) For every formula F either $F \notin T_2$ or $\neg F \notin T_2$. As $T_1 \subseteq T_2$, for any $A \notin T_2$ it necessarily holds that $A \notin T_1$. Therefore T_1 is also consistent.

Exercise 8 **2P+2P=4P**

Given a set $\mathcal{X}$ of propositional formulas, let $\mathsf{Cn}(\mathcal{X})$ denote the set of consequences of $\mathcal{X}$, i.e., the set of all propositional formulas F with $\mathcal{X} \models F$.

Let $\mathcal{X}$ be an arbitrary set of propositional formulas, and $\mathcal{Y} = \{F_1, \ldots, F_n\}$ a finite set of propositional formulas (not necessarily included in $\mathcal{X}$) such that $\mathsf{Cn}(\mathcal{X}) = \mathsf{Cn}(\mathcal{Y})$.

(a) Prove that there is a **finite** subset $\mathcal{X}' \subseteq \mathcal{X}$ such that $\mathsf{Cn}(\mathcal{X}') = \mathsf{Cn}(\mathcal{Y})$ **using the compactness theorem**.

(b) Give an alternative proof for the result of (a) but this time **based on the results regarding the Hilbert calculus**.

Solution: (a) For every F_i the set $\mathsf{Cn}(\mathcal{X}) \cup \{\neg F_i\}$ is unsatisfiable, and so by the compactness theorem it contains a finite unsatisfiable set $C_i \cup \{\neg F_i\}$. It follows $C_i \models F_i$. So $C_1 \cup \ldots \cup C_n \models F_i$ for every i, and so $C_1 \cup \ldots \cup C_n \models F$ for every $F \in \mathsf{Cn}(\mathcal{X})$. Since $C_1 \cup \ldots \cup C_n$ is finite, we are done.

(b) Since $\mathsf{Cn}(\mathcal{X}) = \mathsf{Cn}(\mathcal{Y})$, all formulas of $\mathcal{Y}$ are consequences of $\mathcal{X}$, and so derivable from $\mathcal{X}$ by Hilbert calculus. In the derivations the hypothesis rule is only applied to a finite number of formulas of $\mathcal{X}$; let $\mathcal{X}'$ be this set. Then from $\mathcal{X}'$ we can derive all formulas of $\mathcal{Y}$, and so all formulas of $\mathsf{Cn}(\mathcal{Y})$, and we are done.