Note Title 27.06.2011

Exercise 6.1 **Herbrand universe**

(a) For each of the following formulas, give the Herbrand universe.

i) $P(x) \to P(c)$ $\{c\}$

(1) All constants occurring in F belong to $D(F)$; if no constant occurs in F, then $a \in D(F)$.

(2) For every n-ary function symbol f occurring in F, if $t_1, t_2, \ldots, t_n \in D(F)$ then $f(t_1, t_2, \ldots, t_n) \in D(F)$.

ii) $P(x) \to Q(f(x), g(c))$

$\{c, f(c), g(c), f^n(c), g(f(c)), f(g(c)), \ldots\}$

iii) $\forall x.\exists y.P(x, y)$ $\equiv_S \forall x\, P(x, f(x))$

$\{a, f(a), f(f(a)) \ldots\}$

(b) What properties must hold for a finite formula, such that the resulting Herbrand universe is finite?

no function symbols with arity > 0
(i.e. only constants)

Exercise 5.2 Modelling

- [illegible]
- [illegible] retu
- [illegible] r
- [illegible] retu

$$\forall x s.\ top(push(x,s)) = x \qquad \mathrm{I}$$

$$\forall x s.\ \neg IsEmpty(push(x,s)) \qquad \mathrm{II}$$

$$\forall x s.\ pop(push(x,s)) = s \qquad \mathrm{III}$$

$$\forall s.\ \neg IsEmpty(s) \rightarrow Stack(pop(s)) \qquad \mathrm{IV}$$

$$\forall s.\ \neg IsEmpty(s) \rightarrow \exists x.\ top(s) = x \qquad \mathrm{V}$$

$$Stack(s) := \mathrm{I} \wedge \mathrm{II} \wedge \mathrm{III} \wedge \mathrm{IV} \wedge \mathrm{V}$$

(b) Give a model $\mathcal{A}$ of your formula where the universe $U_{\mathcal{A}}$ contains at least the two distinct objects a, b.

$$U_{\mathcal{A}} \supseteq \{ \underline{a}, \underline{b}, \underline{push(a,b)}, top(a), \underline{top(push(a,b))}, \underline{pop(push(a,b))}, \underbrace{push(push(a,b), push(b,a))}_{*}, \underline{pop(*)} \ldots \}$$

$$U_{\mathcal{A}/=} = \{ a, b, push(a,b), push(push(a,b), a), push(push(a,b), b) \ldots \}$$

$$\approx \{ a, b, (a,b), ((a,b), a), ((a,b), b) \ldots \}$$

$$IsEmpty^{\mathcal{A}} = \{ a, b \}$$

(c) Is the following a model of your formula? $U_{\mathcal{A}} = \mathbb{N}$, $\mathsf{IsEmpty}^{\mathcal{A}} = \{0\}$, $\mathsf{push}^{\mathcal{A}}(x,s) = (2x+1)\cdot 2^s$, $\mathsf{pop}^{\mathcal{A}}(s) = \max\{k \in \mathbb{N} \mid 2^k \text{ divides } s\}$, $\mathsf{top}^{\mathcal{A}}(s) = \max\{0, (s/\underset{2^{\wedge}}{\mathsf{pop}^{\mathcal{A}}(s)} - 1)/2\}$.

$\neg \mathit{IsEmpty}(\mathit{push}(x,s)) \leftrightarrow \underline{(2x+1)}\cdot 2^s \neq 0$ ✓

$\mathit{pop}(\mathit{push}(x,s)) = s \leftrightarrow \max\{k \in \mathbb{N} \mid 2^k \text{ div. } (2x+1)\cdot 2^s\}$

$= s$

odd

$\rightsquigarrow \max k = s$

$\mathit{top}(\mathit{push}(x,s)) = x \leftrightarrow \max\{0, ((2x+1\cdot 2^s / s - 1)$
$/2\}$ ↯

$\rightarrow$ not a model

Exercise 6.2 **Gilmore**

(a) Prove the validity of the following formula using Gilmore's algorithm:

$$(\forall x. P(x, f(x))) \rightarrow (\exists y. P(c, y))$$

Input: F
$n := 0$;
repeat $n := n + 1$;
until $(F_1 \wedge F_2 \wedge \ldots \wedge F_n)$ is unsatisfiable;
report "unsatisfiable" **and halt**.

negate $\longrightarrow$ $\neg[(\forall x\, P(x, f(x))) \rightarrow (\exists y. P(c,y))]$

$\equiv \neg(\neg(\forall x. P(x, f(x))) \vee (\exists y. P(c,y)))$

$\equiv \forall x\, P(x, f(x)) \wedge \forall y \neg P(c,y)$

$\equiv \forall x y.\ P(x, f(x)) \wedge \neg P(c,y)$

$U_A = \{c, f(c), \ldots, f^n(c)\}$

$c/x,\ c/y$: $\underline{P(c, f(c))} \wedge \neg P(c,c) \wedge$

$c/x,\ f(c)/y$: $P(c, f(c)) \wedge \underline{\neg P(c, f(c))}$

(b) Formalize the following propositions in first-order logic and use Gilmore to show that i) implies ii):

i) Professor p is happy if all his students like logic.

ii) Professor p is happy if he has no students.

$H(x)$ ... "x is happy"

$L(x)$ ... "x likes logic"

$S(p,x)$ ... "x is a student of p"

$$F_i \equiv (\forall x\,(S(p,x) \rightarrow L(x)) \rightarrow H(p))$$

$$F_{ii} \equiv (\forall x.\ \neg S(p,x)) \rightarrow H(p)$$

We want to show $F \equiv F_i \rightarrow F_{ii}$

$\rightsquigarrow$ show $\neg F$ is unsat

$$\neg (F_i \to F_{ii})$$

$$\equiv F_i \wedge \neg F_{ii}$$

$$\equiv F_i \wedge \neg (\exists x \; S(p,x) \vee H(p))$$

$$\equiv F_i \wedge (\forall x \; \neg S(p,x)) \wedge \neg H(p)$$

$$\equiv (\forall x \, (S(p,x) \to L(x))) \to H(p)) \wedge (\forall x \; \neg S(p,x) \wedge \neg H(p)$$

$$\equiv ((\forall x \, (\neg S(p,x) \vee L(x))) \to H(p)) \quad \wedge \quad \text{— " —}$$

$$\equiv ((\exists x \; S(p,x) \wedge \neg L(x)) \vee H(p)) \wedge \quad \text{— " —}$$

$$\equiv \exists x \forall y \, ((S(p,x) \wedge \neg L(x)) \vee H(p)) \wedge \neg S(p,y) \wedge \neg H(p))$$

$$\equiv_s \forall y \, ((S(p,a) \wedge \neg L(a)) \vee H(p)) \wedge \neg S(p,y) \wedge \neg H(p))$$

$$\equiv \forall y \, ((S(p,a) \vee H(p)) \wedge (\neg L(a) \vee H(p)) \wedge \neg S(p,y) \wedge \neg H(p))$$

$$U_A = \{a, p\}$$

$a/y \quad (S(p,a) \lor H(p)) \land (\neg L(a) \lor H(p)) \land \neg S(p,a) \land \neg H(p)$

↯ ↯

$\equiv$ False

Exercise 6.4 Ground resolution

$$\forall y.Q(f(a), f(y)) \wedge \forall xy.(Q(y, f(y)) \rightarrow P(f(x), g(y,b))) \\ \rightarrow \exists xyz.(P(x,y) \wedge P(f(a), g(x,b)) \wedge Q(x,z))$$

First step: RNF and negate (and skolemize)

$$\neg(\forall y.\ Q(f(a), f(y)) \wedge \forall xy.(\neg Q(y, f(y)) \vee P(f(x), g(y,b))) \rightarrow \ldots)$$

$$\equiv \neg(\neg(\forall y.\ \ldots) \vee \exists xyz.(P(x,y) \wedge P(f(a), g(x,b)) \wedge Q(x,z)))$$

$$\equiv \forall y.Q(f(a), f(y)) \wedge \forall xy(\neg Q(y, f(y)) \vee P(f(x), g(y,b))) \wedge \forall xyz.(\neg P(x,y) \vee \neg P(f(a), g(x,b)) \vee \neg Q(x,z))$$

$\equiv$ rename

$$\forall y_1.Q(f(a), f(y_1)) \wedge \forall x_2 y_2 (\neg Q(y_2, f(y_2)) \vee P(f(x_2), g(y_2,b))) \wedge \forall x_3 y_3 z_3.(\underline{\neg P(x_3,y_3) \vee \neg P(f(a), g(x_3,b)) \vee \neg Q(x_3,z_3)})$$

$\equiv \forall y_1 x_2 y_2 x_3 y_3 z_3 . \quad \ldots$

$U_A = \{a, b, f(a), g(a,b), g(f(a), a), \ldots\}$

Ground resolution

Clauses: $\{Q(f(a), f(y_1))\}$

$\{\neg Q(y_2, f(y_2)), P(f(x_2), g(y_2, b))\}$

$\{\neg P(x_3, y_3), \neg P(f(a), g(x_3, b)), \neg Q(x_3, z_3)\}$

I step:

$f(a)/y_1$

$f(a)/y_2$

a/x_2

$f(a)/x_3$

$g(a,b)/y_3$

$f(a)/z_3$

$\{Q(f(a), f(f(a)))\}$

$\{\neg Q(f(a), f(f(a))), P(f(a), g(f(a), b))\}$

$\{\neg P(f(a), g(a,b)), \neg P(f(a), g(f(a), b)), \neg Q(f(a), f(a))\}$

$\rightarrow \{P(f(a), g(f(a), b))\}$

$\rightarrow \{\neg Q(f(a), f f(a)), \neg P(f(a), g(a,b)), \neg Q(f(a), f(a))\}$

II step.

a/y_1

$f(a)/y_2$

a/x_2

$f(a)/x_3$

$g(f(a),6)/y_3$

$f(a)/z_3$

$\{Q(f(a), f(a))\}$

$\{\neg Q(f(a), f(f(a))), P(f(a), g(f(a),6))\}$

$\{\neg P(f(a), g(f(a),6)), \neg P(f(a), g(f(a),6)),$

$\neg Q(f(a), f(a))\}$

$\to \{\neg P(f(a), g(f(a), 6))\}$

$\to \{\}$ $\square$

Exercise 6.5 **Fun with equality**

$$\left.\begin{array}{ll} & \forall x(f(x,n) = x) \\ \wedge & \forall x(f(n,x) = x) \\ \wedge & \forall x \forall y \forall z(f(x, f(y,z)) = f(f(x,y),z)) \\ \wedge & a \neq n \\ \wedge & b \neq n \\ \wedge & a \neq b \end{array}\right\} *$$

(a) Transform G into Skolem form. Let H be the resulting formula.

$$\begin{aligned} G \equiv \quad * \quad & \wedge \ \forall x.\,(Eq(x,x)) \\ & \wedge \ \forall xy.\,(Eq(x,y) \to Eq(y,x)) \\ & \wedge \ \forall xyz\,(Eq(x,y) \wedge Eq(y,z) \to Eq(x,z)) \end{aligned}$$

$$\begin{aligned} H \equiv \forall xyz\,(\ & Eq(f(x,n),x) \\ & \wedge \ Eq(f(n,x),x) \\ & \wedge \ Eq(f(x,f(y,z)), f(f(x,y),z)) \\ & \wedge \ \neg Eq(a,n) \wedge \neg Eq(b,n) \wedge \neg Eq(a,b) \\ & \wedge \ \ldots \end{aligned}$$

(b) Determine from the Herbrand expansion $E(H)$ a Herbrand model $\mathcal{A}$ for H, i.e., derive a suitable interpretation $\mathrm{Eq}^{\mathcal{A}}$ over $D(H)$.

$$\mathrm{Eq}^{A} = \{ (a,a), (b,b), (n,n), (f(a,b), f(a,b)),$$
$$(f(a,n),a), (f(n,a),a), (f(n,n),n) \ldots,$$
$$(f(a,f(a,b)), f(f(a,a),b)), \ldots,$$
$$(f(a,n), f(n,a)), (n, f(n,n)) \ldots \}$$

Exercise 6.5 Fun with equality

(c) Construct the equivalence classes $D(H)/\mathrm{Eq}^{A}$.

$$\begin{aligned} &\forall x(f(x,n)=x) \\ \wedge\ &\forall x(f(n,x)=x) \\ \wedge\ &\forall x\forall y\forall z(f(x,f(y,z))=f(f(x,y),z)) \\ \wedge\ &a\neq n \\ \wedge\ &b\neq n \\ \wedge\ &a\neq b \end{aligned}$$

$f(a,b) \approx ab$

$f(a,f(a,b)) =_{Eq} f(f(a,a),b) \approx aab$

$\Rightarrow (a|b|n)^+ \approx (a|b)^*$

The equiv.-classes are $[w]$ for all $w \in \mathcal{L}((a|b)^*)$

$\rightarrow$ all words are in such an equivalence class:

$w = a \rightarrow [a]$

$= b \rightarrow [b]$

$= n \rightarrow [n]$

IS: assume that some word $w \in [w]$

append: $a \to wa \in [wa]$

$b \to wb \in [wb]$

$n \to (wn, w) \in Eq^A \to wn \in [w]$

prepending:

— " —

$\Leftarrow$ need to show $\neg \exists w\, w' \; \forall x \in [w] .\; x \in [w']$

$w \in [w]$

$w' \in [w'] \quad \wedge \quad \neg Eq(w, w')$

character-wise comparison

$\square$

Exercise 6.6 Monadic predicate logic

Monadic predicate logic is predicate logic with the restriction to only use monadic (i.e. arity one) predicates and no function symbols. A formula which fulfills this restriction is called a *monadic formula*.

(a) Show that each monadic formula F is equivalent to a monadic formula G without nested quantors.

$$\forall x \exists y \, (P(x) \vee Q(y)) \equiv \forall x \, P(x) \vee \exists y \, Q(y)$$

assume: $F' = \overline{Q_1 x_1} \ldots \overline{Q_n x_n} \, F^*$
and F^* is in DNF

stepwise approach eliminating quantors from within

two cases for $Q_i x_i$: I) $\forall x_i$ → transform to CNF
→ same as $\exists x_i$

II) $\exists x_i$ → b/c in DNF
push it in front of each clause

for each clause k_i

$$\bigwedge_{m=1}^{l} C_i^m \wedge \bigwedge_{m=1}^{l'} C_i'^m$$

which contain x_i don't contain x_i

$$\equiv \left(\exists x_i \bigwedge_{m=1}^{l} C_i^m\right) \wedge \bigwedge_{m=1}^{l'} C_i'^m$$

(b) Give a decision procedure (i.e. one that always terminates) for deciding whether a given monadic formula F is satisfiable.

1) we don't have function symbols $\longrightarrow$ Herbrand-universe is finite

| 2) a) we can construct formula G w/o nested quant.

b) pull out the quant. again – $\exists$-first

$\exists\exists\exists\forall\forall\forall\, F$ "EA-Fragment"

c) eliminate $\exists$

$\rightarrow$ no functions only constants

3) search the H-universe (it's finite!)

Exercise 6.3 The Drinker Paradox

that there is someone in this pub, that if he is drinking, everyone is drinking.