Logic – Homework 5

Discussed on 20.06.2011.

Exercise 5.1 Tarski's World

(a) Give a Tarski-World that is a model for the following formulas:

$$\forall x\ \exists y\ \neg SameSize(x,y) \tag{1}$$
$$\forall x\ \forall y\ (Triangle(x) \wedge Triangle(y) \wedge SameCol(x,y) \rightarrow SameRow(x,y)) \tag{2}$$
$$\forall x\ \forall y\ (Smaller(x,y) \wedge SameRow(x,y) \rightarrow LeftOf(x,y)) \tag{3}$$
$$Pentagon(b) \wedge Medium(b) \tag{4}$$
$$\forall x\ (Triangle(x) \leftrightarrow \neg SameSize(x,c)) \tag{5}$$
$$\exists x\ \exists y\ (Triangle(x) \wedge Square(y) \wedge Between(x,a,y)) \tag{6}$$
$$\exists x\ \exists y\ (Triangle(x) \wedge Square(y) \wedge SameCol(x,y)) \tag{7}$$
$$Between(b,a,c) \wedge Between(b,c,a) \tag{8}$$
$$\exists y\ (Triangle(y) \vee Pentagon(y) \vee Square(y)) \wedge \forall y. \neg Small(y) \tag{9}$$
$$\forall x\ (Pentagon(x) \rightarrow \exists y.(Pentagon(y) \wedge SameCol(x,y) \wedge \neg SameRow(x,y))) \tag{10}$$

(b) Translate the following propositions into closed first-order formulas in Tarski's World. You are not allowed to use constants.

i) There is no square between any two objects.

ii) The further to left an object is placed, the larger it is.

iii) Nothing is in-between two squares.

iv) A square placed to the right of a triangle is large.

(c) In Tarski's World, state the following predicates:

i) An object is at the rim of the world.

ii) Four objects form a rectangle.

iii) Four objects form an exclamation mark.

iv) Two objects are equal (without using =).

Exercise 5.2 Modelling

A *stack* is an abstract data type which captures the LIFO (last in, first out) principle characterized by the following fundamental operations:

- $\mathsf{IsEmpty}(s)$ evaluates to true iff no element has been added to the stack s.
- $\mathsf{top}(s)$ returns the last element added to s.
- $\mathsf{push}(x,s)$ returns the stack resulting from adding the element x to the stack s.
- $\mathsf{pop}(s)$ returns the stack obtained from s by removing the last element added to s.

(a) Give a formula F in first-order logic with equality using the above function and predicate symbols which formalizes the necessary relations between the operations. For instance, top should always return x for the stack $\mathsf{push}(x,s)$. Note that also a stack can be added as an element to a stack.

(b) Give a model $\mathcal{A}$ of your formula where the universe $U_{\mathcal{A}}$ contains at least the two distinct objects a, b.

(c) Is the following a model of your formula? $U_{\mathcal{A}} = \mathbb{N}$, $\mathsf{IsEmpty}^{\mathcal{A}} = \{0\}$, $\mathsf{push}^{\mathcal{A}}(x,s) = (2x+1)\cdot 2^s$, $\mathsf{pop}^{\mathcal{A}}(s) = \max\{k \in \mathbb{N} \mid 2^k \text{ divides } s\}$, $\mathsf{top}^{\mathcal{A}}(s) = \max\{0, (s/\mathsf{pop}^{\mathcal{A}}(s) - 1)/2\}$.

Exercise 5.3

Show that the following two formulas are semantically equivalent by using the equivalences discussed in the lecture:

$$\begin{aligned} F &= \forall x \forall y\ (\neg P(x, f(z)) \vee (\neg Q(y) \wedge \forall y\ R(y)))) \\ G &= \neg \exists x \exists u \exists y\ ((\neg R(u) \wedge P(x, f(z))) \vee (Q(y) \wedge P(x, f(z)))). \end{aligned}$$

Exercise 5.4

Transform the following formula into Skolem normalform. Explicity state in every step of the transformation if the semantical equivalence is preserved or if only equivalence up to satisfiability holds:

$$F = \forall y \neg ((P(b, g(x)) \vee \forall x Q(f(x))) \wedge R(y)).$$

Exercise 5.5

Give a formula (w/o equality) F such that

(a) every model of F has at least three elements.

(b) every model of F has infinitely many elements.

Exercise 5.6

Often, one wants to express that there exists exactly one object in a given set P which satisfies a FOL formula (with equality) $G(x)$ (with x the only free variable in G). A common abbreviation is $\exists! x \in P \colon G(x)$.

(a) Give a formal definition of this "unique existential quantifier" $\exists!$ using FOL with equality.

(b) Compute for $\neg \exists! x \in P \colon G(x)$ a semantically equivalent rectified formula in prenex form.

Exercise 5.7

Let F be a formula of first-order logic (FOL). Define a translation T such that

- $\mathsf{T}(F)$ is a FOL formula **without function symbols and without equality**.
- F is satisfiable if and only if $\mathsf{T}(F)$ is satisfiable.

Explicitly apply your translation to the following formula:

$$\exists x \forall y (f(x, y) = f(y, x)) \wedge \forall x \forall y \forall z (f(x, f(y, z)) = f(f(x, y), z))$$

Exercise 5.8 *

Give a finite model for the following formula:

$$\begin{aligned} F \quad := \quad & \forall p \in P \colon \forall q \in P \colon (p \neq q \rightarrow \exists! g \in G \colon (g = l(p, q))) \\ \wedge \quad & \forall g \in G \colon \forall h \in G \colon (g \neq h \rightarrow \exists! p \in P \colon (p = i(g, h)) \\ \wedge \quad & \forall g \in G \colon \exists p \in P \colon \exists q \in P \colon \exists r \in P \colon (g = l(p, q) \wedge g = l(q, r) \wedge g = l(r, p) \wedge p \neq q \wedge q \neq r \wedge r \neq p) \\ \wedge \quad & \exists g \in G \colon \exists h \in G \colon (g \neq h) \\ \wedge \quad & \forall x (P(x) \leftrightarrow \neg G(x)). \end{aligned}$$