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Logic – Homework 4

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Exercise 4.1

(a) Which of the following formulas is closed? State all formulas again with unique variable names.

i) $\exists y.Q(y,y) \rightarrow (\forall y.P(y) \wedge R(y)) \vee R(y)$

ii) $\exists y.Q(y,y) \rightarrow (\forall y.P(y) \wedge R(y)) \vee R(x)$

iii) $(\exists y.Q(y,y)) \rightarrow (\forall y.P(y) \wedge R(y)) \vee R(y)$

iv) $Q(y,y) \rightarrow (\forall y.P(y) \wedge R(y)) \vee \exists y.R(y)$

v) $\forall y.Q(y,y) \rightarrow (\forall y.P(y) \wedge R(y)) \vee \exists y.R(y)$

(b) Which of the following formulas is valid, which is satisfiable?

For each satisfiable formula give a model. For each model that is satisfiable, but not valid, give a counter-model, i.e. a model where the formula is not satisfied.

i) $(\forall x.\exists y.P(x,y)) \rightarrow (\exists y.\forall x.P(x,y))$

ii) $(\exists x.\forall y.P(x,y)) \rightarrow (\forall y.\exists x.P(x,y))$

iii) $(\exists x.\forall y.P(x,y)) \rightarrow (\forall x.\exists y.P(x,y))$

(c) Which of the following structures is a model for the formula F?

$$F = \exists x \exists y \exists z.(P(x,y) \wedge P(z,y) \wedge P(x,z) \wedge \neg P(z,x))$$

i) $U_{\mathcal{A}} = \mathbb{N}, P^{\mathcal{A}} = \{(m,n) \mid m,n \in \mathbb{N}, m < n\}$

ii) $U_{\mathcal{A}} = \mathbb{N}, P^{\mathcal{A}} = \{(m,m+1) \mid m \in \mathbb{N}\}$

iii) $U_{\mathcal{A}} = 2^{\mathbb{N}}, P^{\mathcal{A}} = \{(A,B) \mid A,B \subseteq \mathbb{N}, A \subseteq B\}$

Exercise 4.2 Formal properties

(a) Prove that a formula containing only $\wedge, \vee, \forall, \exists, \rightarrow$ and atomic formulas is always satisfiable.

(b) Give the definition of a function $FV()$, which returns the set of free variables of a formula. Use induction on the structure of the formula.

Exercise 4.3

In mathematics and computer science you will quite often find propositions like

"There is an $x \in M$, such that $P(x)$ holds." or (formally) "$\exists x \in M\colon P(x)$"

But the syntax of predicate logic shown in the lecture does not allow a construct like $\exists x \in M$.

(a) Find a way of expressing such a proposition in predicate logic.

(b) Do the same for the all-quantification: "$\forall x \in M\colon P(x)$".

(c) Show that your translation still honors the following equivalence:

$$\neg \exists x \in M\colon P(x) \equiv \forall x \in M\colon \neg P(x)$$

Exercise 4.4 Tarski's World

(a) Give a Tarski-World that is a model for the following formulas:

$$\forall x \exists y. \neg SameSize(x, y) \tag{1}$$
$$\forall x \forall y. (Triangle(x) \land Triangle(y) \land SameCol(x, y) \rightarrow SameRow(x, y)) \tag{2}$$
$$\forall x \forall y. (Smaller(x, y) \land SameRow(x, y) \rightarrow LeftOf(x, y)) \tag{3}$$
$$Pentagon(b) \land Medium(b) \tag{4}$$
$$\forall x. (Triangle(x) \leftrightarrow \neg SameSize(x, c)) \tag{5}$$
$$\exists x \exists y. Triangle(c) \land Square(y) \land Between(x, a, y) \tag{6}$$
$$\exists x \exists y. Triangle(c) \land Square(y) \land SameCol(x, y) \tag{7}$$
$$Between(b, a, c) \land Between(b, c, a) \tag{8}$$
$$(\exists y. Triangle(y) \lor Pentagon(y) \lor Square(y)) \land \forall y. \neg Small(y) \tag{9}$$
$$\forall x. (Pentagon(x) \rightarrow \exists y. (Pentagon(y) \land SameCol(x, y) \land \neg SameRow(x, y))) \tag{10}$$

(b) Translate the following propositions into closed first-order formulas in Tarski's World. You are not allowed to use constants.

i) There is no square between any two objects.

ii) The further to left an object is placed, the larger it is.

iii) Nothing is in-between two squares.

iv) A square placed to the right of a triangle is large.

(c) In Tarski's World, state the following predicates:

i) An object is at the rim of the world.

ii) Four objects form a rectangle.

iii) Four objects form an exclamation mark.

iv) Two objects are equal (without using =).

Exercise 4.5 Modelling

Recall the definition of a *stack*: An abstract data structure, where elements can be pushed onto and then be popped from on a last-in first-out basis.

State, as formulas in first-order predicate logic, the necessary relations between the three functions $pop(S), top(S)$ and $push(x, S)$, and the predicate $Empty(S)$, such that they describe the known behavior of a stack. As an example, the constraint $\forall x \forall S. \neg Empty(push(x, S))$ must hold.

As a reminder: The descriptions of the functions and predicates:

$pop(S)$ pops the top-most element from the stack S and returns the resulting stack.

$top(S)$ returns the top-most element of the stack S (without popping it!).

$push(x, S)$ pushes the element x onto the stack S and return the resulting stack.

$Empty(S)$ returns true iff the stack S is empty.

Exercise 4.6

Let $\mathbf{M}$ be a set of propositional formulas, and F a propositional formula. Show:

$$\mathbf{M} \models F \text{ iff there exists a finite subset } \mathbf{N} \subseteq \mathbf{M} \text{ such that } \mathbf{N} \models F.$$

Remark: $\mathbf{M} \models F$ is defined by: for any assignment $\mathcal{A}$ suitable for all formulas in $\mathbf{M} \cup \{F\}$, if $\mathcal{A}$ is a model for all formulas in $\mathbf{M}$, then it is also a model for F.

Exercise 4.7 *

This exercise is meant to explain where the name compactness *theorem comes from. It won't be discussed in the tutorials as it is not relevant for the exam. Feel free to mail us in case of questions or if you want us to check your solutions.*

Some definitions of basic topology:

- A *topological space* $(X, \mathcal{T})$ is a set X together with a collection (=set) $\mathcal{T}$ of subsets of X ($\mathcal{T} \subseteq 2^X$). $\mathcal{T}$ is called a *topology* of X and its elements $O \in \mathcal{T}$ are called the *open sets* of X where $\mathcal{T}$ has to satisfy the following axioms:

  A1 $\emptyset$ and X are open, i.e., $\{\emptyset, X\} \subseteq \mathcal{T}$.

  A2 The intersection of a finite number of open sets is open, i.e., for any $\mathcal{C} \subseteq \mathcal{T}$ with $|\mathcal{C}| < \infty$, also $\bigcap_{O \in \mathcal{C}} O \in \mathcal{T}$.

  A3 The union of an arbitrary number of open sets is open, i.e., for any $\mathcal{C} \subseteq \mathcal{T}$ also $\bigcup_{O \in \mathcal{C}} O \in \mathcal{T}$.

  A set $A \subseteq X$ is *closed* if its complement $\overline{A} = X \setminus A$ is open.

- A more convenient way of representing a topology $\mathcal{T}$ is (sometimes) by means of a *base*:

  For any collection $\mathcal{C} \subseteq 2^X$ of subsets of X, let $\mathsf{T}(\mathcal{C}) := \{\bigcup_{O \in \mathcal{C}} O \mid \mathcal{C} \subseteq \mathcal{T}\}$ (with $\emptyset = \bigcup_{O \in \emptyset} O$). If $\mathsf{T}(\mathcal{C})$ is a topology, we call $\mathcal{C}$ a *base* of $\mathsf{T}(\mathcal{C})$ and say that $\mathcal{C}$ *generates* the topology $\mathsf{T}(\mathcal{C})$.

  Every topology $\mathcal{T}$ has a base, namely itself, as $\mathcal{T} = \mathsf{T}(\mathcal{T})$ by A3. But not any collection $\mathcal{C} \subseteq 2^X$ needs to generate a topology, e.g., $\mathcal{C} = \emptyset$ for $X \neq \emptyset$.

- A (finite) collection $\mathcal{C} \subseteq \mathcal{T}$ of open sets is a *(finite) open cover* of X if $X = \bigcup_{O \in \mathcal{C}} O$. $(X, \mathcal{T})$ is *compact* if *every* open cover $\mathcal{C}$ of X contains already a finite open cover $\mathcal{C}' \subseteq \mathcal{C}$ ($|\mathcal{C}'| < \infty$).

- *Example*: The standard topology of the reals $\mathbb{R}$ is generated by the collection of all open intervals. You might remember that a set $A \subseteq \mathbb{R}$ is usually called compact if it is *closed* and *bounded*. From the *Heine-Borel theorem* it follows that for the reals this definition is equivalent to the above definition.

- *Example*: For every set X the *discrete topology* is simply given by 2^X and generated by $\{\{x\} \mid x \in X\}$. $(X, 2^X)$ is compact iff X is finite as $\{\{x\} \mid x \in X\}$ is an open cover.

- *Example*: For every set X the *trivial topology* is given by $\{\emptyset, X\}$ and generated by $\{X\}$. X is always compact w.r.t. the trivial topology.

- *Example*: Every finite topological space $(X, \mathcal{T})$ ($|X| < \infty$) is compact.

Let $\mathbb{B}$ denote the set $\{0, 1\}$, and for any set S let $\mathbb{B}^S$ denote the set of all functions $f : S \to \mathbb{B}$. The standard topology on $\mathbb{B}$ is the discrete topology and there is a canonical construction (a particular instance of the *product topology*) for extending it to $\mathbb{B}^S$ for any set S:

For any $c \in \mathbb{B}^I$ with I some *finite* subset of S, its *cylinder set* $[c] \in \mathbb{B}^S$ consists of all possible extensions of c to a function from S to $\mathbb{B}$:

$$[c] := \{f \in \mathbb{B}^S \mid \forall i \in I \colon f(i) = c(i)\}.$$

Let $\mathcal{Z}_S$ denote the collection of all cylinder sets in $\mathbb{B}^S$. Then $\mathcal{T}_S = \mathsf{T}(\mathcal{Z}_S)$ is the product topology on $\mathbb{B}^S$ (generated by the discrete topology on $\mathbb{B}$).

(a) Show that $(\mathbb{B}^{\mathbb{N}}, \mathcal{T}_{\mathbb{N}})$ is indeed a topological space, i.e., show that $\mathcal{T}_{\mathbb{N}}$ satisfies the axioms A1, A2, A3.

(b) Show that $(\mathbb{B}^{\mathbb{N}}, \mathcal{T}_{\mathbb{N}})$ is compact by means of the compactness theorem.

  Remark: Every $f \in \mathbb{B}^{\mathbb{N}}$ induces an assignment $\mathcal{A}_f$ in the natural way: $\mathcal{A}_f(A_i) := f(i)$. Therefore, simply write $f \models F$ for $\mathcal{A}_f \models F$. Show that any formula F is satisfiable iff there is some $f \in \mathbb{B}^{\mathbb{N}}$ with $f \models F$.

(c) In the definition of propositional formula, we only consider a countable set of atomic formulas $\{A_i \mid i \in \mathbb{N}\}$. Assume we take as atomic formulas the set $\{A_x \mid x \in \mathbb{R}\}$ instead. Then the proof of the compactness theorem given in the slides does not work anymore (why?). But by *Tychonoff's theorem* $(\mathbb{B}^{\mathbb{R}}, \mathcal{T}_{\mathbb{R}})$ is still compact (assuming the axiom of choice).

  Deduce from this that the compactness theorem holds also for propositional formulas constructed from the atomic formulas $\{A_x \mid x \in \mathbb{R}\}$.