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Fundamental Algorithms

Solution Keys 1

1. We prove the correctness of the multiplication *à la russe* by mathematical induction. Let $P(m,n)$ be the multiplication result of positive integers m and n.

   Basis Obviously, $P(1,n) = n$ for any positive integers n.

   Inductive step From the definition of $P(m,n)$ we have

   $$P(m,n) = P(\left\lfloor \frac{m}{2} \right\rfloor, 2n) + x \ ,$$

   where $x = 0$, if m is even and $x = n$, if m is odd. Since

   $$P(\left\lfloor \frac{m}{2} \right\rfloor, 2n) = \left\lfloor \frac{m}{2} \right\rfloor \cdot 2n$$

   by the induction hypothesis. It follows that $P(m,n) = m \cdot n$.

2. See exercise sheet 2.

3. (a) True. Choose $c = 10$ and $n_0 = 2$. Then, for all $n \geq 2$:

   $$10n + 40 \leq 10n^2 + 20n + 10 \ .$$

   (b) True. Choose $c_1 = 1$, $c_2 = \sqrt{2}$, and $n_0 = 23$. Then, for all $n \geq 23$:

   $$\begin{array}{c} -23 \leq 0 \quad \text{and} \quad 23 \leq n \\ -23 \leq (1-1)n \quad \text{and} \quad 23 \leq (2-1)n \\ -23 \leq (1-c_1^2)n \quad \text{and} \quad 23 \leq (c_2^2-1)n \\ c_1^2 n \leq n + 23 \quad \text{and} \quad n + 23 \leq c_2^2 n \\ c_1^2 n \leq n + 23 \leq c_2^2 n \\ c_1\sqrt{n} \leq \sqrt{n+23} \leq c_2\sqrt{n} \ . \end{array}$$

   (c) False. We need to prove that $\forall c > 0 \ \forall n_0 \ \exists n \geq n_0 : n^{1000} \leq c \cdot 2^n$, i.e. we need to show that there always exists n that makes the inequality true for any choice of c and n_0. Choose $n = \max(n_0, 14000 - \log_2 c)$. We have $n \geq 14000 - \log_2 c$, which implies

   $$\begin{array}{rcl} n + \log_2 c & \geq & 1000 \log_2 n \\ c \cdot 2^n & \geq & n^{1000} \ . \end{array}$$

   (d) True. $f(n) = o(g(n))$ if and only if $\forall c > 0 \ \exists n_0 \ \forall n \geq n_0 : \frac{f(n)}{g(n)} < c$, which is exactly the definition of the limit statement $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$.

   (e) True. Recall that $f(n) = \Theta(g(n))$ if and only if $f(n) = \mathcal{O}(g(n))$ and $f(n) = \Omega(g(n))$.

i. We first show that if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = L$, where $0 < L < \infty$, then $f(n) = \mathcal{O}(g(n))$. From the definition of the limit, we have

$$\forall \epsilon > 0 \; \exists n_0 \; \forall n \geq n_0 : \left| \frac{f(n)}{g(n)} - L \right| < \epsilon \; .$$

Therefore, if $\epsilon = 1$, there must exist an n_0 such that for all $n \geq n_0$:

$$\begin{aligned} \left| \frac{f(n)}{g(n)} - L \right| &< 1 \\ \frac{f(n)}{g(n)} &< L + 1 \\ f(n) &< (L+1) \cdot g(n) \; . \end{aligned}$$

Now choose $c = L + 1$ in the definition of $\mathcal{O}$ to prove that $f(n) = \mathcal{O}(g(n))$.

ii. Next, we show that if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = L$, where $0 < L < \infty$, then $f(n) = \Omega(g(n))$. From the limit statement, we have $\lim_{n\to\infty} \frac{g(n)}{f(n)} = \frac{1}{L}$. Since $0 < 1/L < \infty$, it follows that $g(n) = \mathcal{O}(f(n))$, which in turn implies $f(n) = \Omega(g(n))$.