Complexity Theory

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Lecture 7

Hierarchies

Agenda

- deterministic time hierarchy theorem
- non-deterministic time hierarchy theorem
- space hierarchy theorem
- relation between space and time

Time Hierarchy Theorem

Theorem (Time Hierarchy)

Let $f, g : \mathbb{N} \to \mathbb{N}$ be time-constructible such that $f \cdot \log f \in o(g)$. Then **DTIME**$(f(n)) \subset$ **DTIME**$(g(n))$.

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- inclusion is strict
- proof: diagonalization
  - TM D simulates M_x on x for $g(|x|)/\log(|x|)$ steps and flips any answer
  - D runs in $\mathcal{O}(g)$
  - if computable by $E = M_i$ in $\mathcal{O}(f)$ then $D(i) \neq M_i(i) = E(i)$, contradiction

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- logarithmic factor due to slowdown in universal simulation
- shows that **P** does not collapse to level k
- corollary: $\mathbf{P} \subset \mathbf{EXP}$

Non-deterministic versions

Theorem (Time Hierarchy (non-det))

Let $f, g : \mathbb{N} \to \mathbb{N}$ be time-constructible such that $f(n+1) \in o(g(n))$. Then $\mathbf{NTIME}(f(n)) \subset \mathbf{NTIME}(g(n))$.

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- inclusion is strict
- proof by lazy diagonalization (see: AB Th. 3.2)
- note: proof of deterministic theorem does not carry over

Space Hierarchy Theorem

Theorem (Space Hierarchy)

Let $f, g : \mathbb{N} \to \mathbb{N}$ be space-constructible such that $f \in o(g)$. Then **SPACE**$(f(n)) \subset$ **SPACE**$(g(n))$.

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- inclusion is strict
- stronger theorem than corresponding time theorem
  - universal TM for space-bounded computation incurs only constant space overhead
  - f, g can be logarithmic too
- proof analogous to deterministic time hierarchy
- corollary: **L** $\subset$ **PSPACE**

Agenda

- deterministic time hierarchy theorem ✓
- non-deterministic time hierarchy theorem ✓
- space hierarchy theorem ✓
- relation between space and time

Relation between time and space

Theorem (Time vs. Space)

Let $s : \mathbb{N} \to \mathbb{N}$ *be space-constructible. Then*

$$\textbf{DTIME}(s(n)) \subseteq \textbf{SPACE}(s(n)) \subseteq \textbf{NSPACE}(s(n)) \subseteq \textbf{DTIME}(2^{O(s(n))})$$

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- inclusions are non-strict
- first two are obvious
- third inclusion requires notion of configuration graphs
- first inclusion can be strengthened to $\mathbf{DTIME}(s(n)) \subseteq \mathbf{SPACE}(\frac{s(n)}{\log n})$

Configuration Graphs

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Let M be a deterministic or non-deterministic TM using $s(n)$ space. Let x be some input.

- this induces a configuration graph $G(M, x)$
- nodes are configuration
  - states
  - content of work tapes
- edges are transitions (steps) that M can take

Properties of configuration graph

- outdegree of $G(M, x)$ is 1 if M is deterministic; 2 if M is non-deterministic
- $G(M, x)$ has at most $|Q| \cdot \Gamma^{c \cdot s(n)}$ nodes (c some constant)
- which is in $2^{\mathcal{O}(s(n))}$
- $G(M, x)$ can be made to have unique source and sink
- acceptance $\sim$ existence of path from source to sink
- which can be checked in time $\mathcal{O}(G(M, x))$ using BFS

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$\Rightarrow$ **NSPACE**$(s(n)) \subseteq$ **DTIME**$(2^{\mathcal{O}(s(n))})$

$\Rightarrow$ **DTIME**$(s(n)) \subseteq$ **NTIME**$(s(n)) \subseteq$ **SPACE**$(s(n))$

- configurations include a counter over all possible choices

References

- the proof of Ladner's theorem given here follows AB, Th. 3.3
- nice survey, see `blog.computationalcomplexity.org/media/ladner.pdf`
- original proof of time hierarchy by *Hartmanis and Stearns* On the computational complexity of algorithms in Transactions of the American Mathematical Society 117.
- non-det time hierarchy by *Stephen Cook*: A hierarchy for nondeterministic time complexity in 4th annual ACM Symposium on Theory of Computing.
- stronger result on time vs space using pebble games by *Hopcroft, Paul, and Valiant* On time versus space in Journal of the ACM 24(2):332-337, April 1977.

Summary

- a lot of diagonalization
- Ladner: **NP**-intermediate languages exist
- $f \cdot \log f \in o(g)$ implies **DTIME**$(f(n)) \subset$ **DTIME**$(g(n))$
- $f \in o(g)$ implies **SPACE**$(f(n)) \subset$ **SPACE**$(g(n))$
- **DTIME**$(f(n)) \subseteq$ **SPACE**$(s(n)) \subseteq$ **NSPACE**$(s(n)) \subseteq$ **DTIME**$(2^{O(s(n))})$
- **P** $\subset$ **EXP** and **L** $\subset$ **PSPACE**

Next time: **PSPACE**