Complexity Theory

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Lecture 5

NP-completeness (2)

Agenda

- Cook-Levin
- SAT demo
- see old friends
  - 0/1−ILP
  - Indset
  - 3−Coloring

Cook-Levin: 3SAT is NP-complete

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  - choose $L \in$ **NP** arbitrary, $L \subseteq \{0,1\}^*$
  - find reduction f from L to 3SAT
    - $\forall x \in \{0,1\}^*$: $x \in L \Leftrightarrow f(x) \in$ 3SAT i.e. φ_x is satisfiable
    - f is polynomial time computable

TMs for L and f

$L \in$ **NP** iff there exists a TM M that runs in time T and there is a polynomial p such that

$$\forall x \in L \ \exists u \in \{0,1\}^{p(|x|)} \ M(x,u) = 1 \Leftrightarrow x \in L$$

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Assumptions

- fix $n \in \mathbb{N}$ and $x \in \{0,1\}^n$ arbitrary
- $m = n + p(n)$
- $M = (\Gamma, Q, \delta)$
- M is oblivious
- M has two tapes
- define TM M_f that takes M, T, p, x and outputs φ_x

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It does all this in time polynomial in n!

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- "computation variables"

$$\begin{array}{lllll} z_1 & z_2 & \ldots & z_{c-1} & z_c \\ z_{c+1} & z_{c+2} & \ldots & z_{2c-1} & z_{2c} \\ \vdots & & & & \vdots \\ z_{c(T(m)-1)+1} & & & & z_{cT(m)} \end{array}$$

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  - each row a snapshot
  - needs $c - 2$ bits to encode state q (independent of x) and 2 bits for the symbols read
- φ_x means "computation on the input is accepting"

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- first snapshot s_1 is $\langle q_{start}, \triangleright, \triangleright \rangle$
- last snapshot $s_{T(m)}$ has state q_{halt} and ouputs 1
- s_{i+1} computed correctly from
  - δ
  - s_i
  - $y_{inputpos(i+1)}$
  - $s_{prev(i+1)}$

$\varphi_x = \varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \varphi_4$

1. relate x and $y_1, \ldots, y_m$: $\bigwedge_{1 \leq i \leq n} x_i = y_i$, where
   $x = y \Leftrightarrow (x \vee \overline{y}) \wedge (\overline{x} \vee y)$

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1. relate x and $y_1, \ldots, y_m$: $\bigwedge_{1 \le i \le n} x_i = y_i$, where $x = y \Leftrightarrow (x \lor \overline{y}) \land (\overline{x} \lor y)$
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Polynomial in n!

Stop!

- $|\varphi_x|$ polynomial in n
- if φ_x is satisfiable, the satisfying assignment yields certificate $y_{n+1}, \ldots y_{n+p(n)}$
- if a certificate exists in $\{0, 1\}^{p(n)}$, we get a satisfying assignment
- M_f can output φ_x in polynomial time

$\Rightarrow$ reduction

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$\Rightarrow$ reduction

- but: not to 3SAT

From CNF to 3CNF

As a last polynomial step, M_f applies the following transformation for each clause

$$u_1 \vee u_2 \vee \ldots \vee u_k$$
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$$u_1 \lor u_2 \lor \ldots \lor u_k$$

$$\leadsto$$

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Each clause with k variables transformed into equivalent $k-2$ 3-clauses with $2k-2$ variables. All x_i fresh.

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Example. $x \vee \overline{y} \vee \overline{z} \vee w$ becomes $x \vee \overline{y} \vee q$ and $\overline{q} \vee \overline{z} \vee w$.

What you need to remember

- for each $L \in$ **NP** take TM M deciding L in polynomial time
- define TM M_f computing a reduction to formula φ_x for each input
- due to obliviousness M_f pre-computes head positions and every computation takes time $T(n + p(n))$ steps
- and is a sequence of snapshots $\langle q, 0, 1 \rangle$
- φ has four parts
  - correct input x, u with u being the certificate
  - correct starting snapshot
  - correct halting snapshot
  - how to go from s_i to s_{i+1}
- finally: CNF transformed to 3CNF

Agenda

- Cook-Levin ✓
- SAT demo
- see old friends
  - $0/1-$ILP
  - Indset
  - $3-$Coloring

So 3SAT is intractable?

- if **P** $\neq$ **NP**, no polynomial time algorithm for SAT
- contrapositive: if you find one, you prove **P** $=$ **NP**
- every problem in **NP** solvable by exhaustive search for certificates
- which implies **NP** $\subseteq$ **PSPACE** (try each possible re-using space)

SAT **is easy!**

- well-researched problem
- has its own conference
- 1000s of tools, academic and commercial
- extremely useful for modelling
    - verification
    - planning and scheduling
    - AI
    - games (Sudoku!)
- useful for reductions due to low combinatorial complexity
- `satlive.org`: solvers, jobs, competitions

Demo

- `www.sat4j.org`
- two termination problems from string/term-rewriting
- 10000s of variables, millions of clauses
- solvable in a few seconds!

Agenda

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More reductions from 3SAT

We will now describe reductions from 3SAT to

- 0/1−ILP: the set of satisfiable sets of integer linear programs with boolean solutions
- Indset $= \{\langle G, k\rangle \mid G$ has independent set of size at least $k\}$
- 3−Coloring $= \{G \mid G$ is 3-colorable$\}$

This establishes **NP**-hardness for all of the problems. Of course, they are easily in **NP** as well, hence complete.

3SAT $\leq_p$ 0/1−ILP

$$(x \vee \overline{y} \vee z) \wedge (x \vee \overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{y} \vee w) \wedge (\overline{x} \vee y \vee \overline{w})$$

$3SAT \le_p 0/1\text{–}ILP$

$$(x \vee \overline{y} \vee z) \wedge (x \vee \overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{y} \vee w) \wedge (\overline{x} \vee y \vee \overline{w})$$

$$\begin{aligned} x + (1-y) + z &\geq 1 \\ x + (1-y) + (1-z) &\geq 1 \\ (1-x) + (1-y) + w &\geq 1 \\ (1-x) + y + (1-w) &\geq 1 \end{aligned}$$

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- $f(x) = x$
- $f(\overline{x}) = (1-x)$
- $f(u_1 \vee \ldots \vee u_k) = f(u_1) + \ldots + f(u_k) \geq 1$

3SAT $\leq_p$ 0/1−ILP

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- $f(x) = x$
- $f(\overline{x}) = (1 - x)$
- $f(u_1 \vee \ldots \vee u_k) = f(u_1) + \ldots + f(u_k) \geq 1$
- linear reduction
- φ satisfiable iff $f(\varphi)$ has boolean solution

3SAT $\leq_p$ Indset

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3SAT $\leq_p$ Indset

- given: formula φ with m clauses of form $C_i = u_{i1} \vee u_{i2} \vee u_{i3}$
- reduce to graph $G = (V, E)$, such that each clause gets a node per satisfying assignment
  - $V = \{C_i^{a_i} \mid a : vars(C_i) \rightarrow \{0, 1\}, C_i \text{ holds under assignment } a_i\}$

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- edges denote conflicting assignments
  - $E = \{\{C_i^{a}, C_{i'}^{a'}\} \mid i, i' \in [m], \exists x.a(x) \neq a'(x)\}$

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- edges denote conflicting assignments
  - $E = \{\{C_i^a, C_{i'}^{a'}\} \mid i, i' \in [m], \exists x.a(x) \neq a'(x)\}$
- G has $7m$ nodes and $\mathcal{O}(m^2)$ edges and can be computed in polynomial time

3SAT $\leq_p$ Indset

- φ is satisfiable

$\Rightarrow$ exists assignment $a : X \rightarrow \{0, 1\}$ that makes φ true

$\Rightarrow$ a makes every clause true

$\Rightarrow$ $\{C_i^{a|vars(i)} \mid 1 \leq i \leq m\}$ is an independent set of size m

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- G has an independent set of size m

$\Rightarrow$ ind. set covers all clauses

$\Rightarrow$ ind. set yields composable, partial assignments per clause

$\Rightarrow$ φ is satisfiable

3SAT $\leq_p$ 3−Coloring

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3SAT $\leq_p$ 3−Coloring

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- reduce to graph $G = (V, E)$
- V is the union of
  - $X \cup \overline{X}$ to capture assignments
  - special nodes $\{u, v\}$
  - one little house per clause with 5 nodes: $\{v_{ij}, a_i, b_i \mid i \in [m], j \in [3]\}$

3SAT $\leq_p$ 3−Coloring

- given: formula φ with m clauses of form $C_i = u_{i1} \lor u_{i2} \lor u_{i3}$
- reduce to graph $G = (V, E)$
- V is the union of
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  - special nodes $\{u, v\}$
  - one little house per clause with 5 nodes: $\{v_{ij}, a_i, b_i \mid i \in [m], j \in [3]\}$
- E comprised of
  - edge $\{u, v\}$
  - for each literal in each clause, a connection to the assignment graph: $\{\{u_{ij}, v_{ij}\} \mid i \in [m], j \in [3]\}$
  - house edges: $\{\{v, a_i\}, \{v, b_i\}, \{v_{i1}, a_i\}, \{v_{i1}, b_i\}, \{v_{i2}, a_i\}, \{v_{i3}, b_i\}, \{v_{i2}, v_{i3}\} \mid i \in [m]\}$
- G has $2n + 5m + 2$ nodes and $\mathcal{O}(m^2)$ edges and can be computed in polynomial time
- three colors: $\{red, true, false\}$

3SAT $\leq_p$ 3−Coloring

- φ is satisfiable,

$\Rightarrow$ there is an assignment $a : X \rightarrow \{0, 1\}$ that makes every clause true

$\Rightarrow$ coloring u red, v false, and x true iff $a(x) = 1$ leads to a correct 3-coloring

3SAT $\leq_p$ 3−Coloring

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$\Rightarrow$ coloring u red, v false, and x true iff $a(x) = 1$ leads to a correct 3-coloring

- G is 3-colorable
- wlog. assume u is red and v is false
- assume there is a clause j such that all literals are colored false

$\Rightarrow$ v_{j2} and v_{j3} are colored true and red

$\Rightarrow$ a_j and b_j are colored true and red

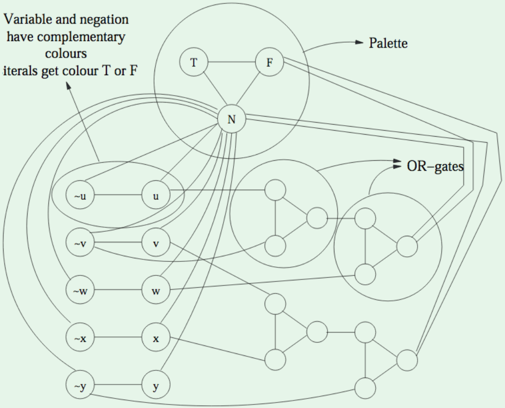
$\Rightarrow$ v_{j1} colored false, which is a contradiction, because it is connected to a false literal

3SAT $\leq_p$ 3−Coloring

Alternatively:

Example

$$\varphi = (u \vee \neg v \vee w) \wedge (v \vee x \vee \neg y)$$



What have you learnt?

- SAT is **NP**-complete
- SAT is practically feasible
- SAT has lots of academic and industrial applications
- SAT can be reduced to independent set, 3-coloring and boolean ILP, which makes those **NP**-hard
- up next: **coNP**, Ladner