Complexity Theory

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Lecture 24’

$AC^0 \subset NC^1$: original proof

(Furst-Saxe-Sipser 1984)

Agenda

Tool: still random assignments

Separate arguments for *wide* and *narrow* conjunctions/disjunctions

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Separate arguments for *wide* and *narrow* conjunctions/disjunctions

- circuits to trees
- make bottom layer fan-in bounded
- make bottom *two-layer* subtrees bounded
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Paths from root: just a polynomial number (depth is fixed)
Copy subgraphs as needed until we get trees
Same depth, still polynomial size

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Depth: number of conjunction/disjunction layers

Assume we have a sequence of minimal depth

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Bounded bottom layer fan-in

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Estimate for a single operation
Separately wide ($\geqslant c \log n$) and narrow cases

Bottom layer: cases

Wide:
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$c = 8k$, only n^k sources of problems, union bound

Bottom layer: result

Probability of $< \sqrt{n}/2$ assignments of $*$ is also small
By union bound: we still have optimal depth, worse polynomial size

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Reassign k

We have minimal-depth n^k-sized tree circuits for parity with fan-in c in the bottom layer

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Goals:

- $\geqslant \sqrt{n}/2$ variables left free ($*$)
- all layer-two operations depend on $b(c)$ variables

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Set of their inputs: H
H hits each argument node
Fixing values of $*$ in H: by induction, dependency on $b(c - 1)$ inputs

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H hits each argument node
Fixing values of $*$ in H: by induction, dependency on $b(c-1)$ inputs
$|H| \leqslant b(c)c \log n$; Probably $< 4k$ entries of $*$; dependency on
$4k + 2^{4k} b(c-1)$ is OK.

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Depth reduction

Second layer elements depend on fixed number of inputs — brute force CNF/DNF, polynomial blowup, lower depth

Contradiction!

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