Complexity Theory

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Lecture 21 NP \subseteq PCP[poly(n), 1]

Recap: Two views of the PCP theorem

prob. checkable proofs		hardness of approximation
PCP verifier V	\leftrightarrow	CSP instance
proof π	\leftrightarrow	variable assignment
π	\leftrightarrow	number of variables in CSP
number of random bits	\leftrightarrow	log <i>m</i> , where <i>m</i> is number of clauses
number of queries	\leftrightarrow	arity of constraints

Goal and plan

Goal

- proof a weaker PCP theorem
- · learn interesing encoding/decoding schemes useful in such proofs

Plan

- proof
 - an NP-complete language: Quadeq
 - Walsh-Hadamard encodings
 - a PCP[poly, 1] system for Quadeq
- summary: PCP and hardness of approximation

Weak PCP

Theorem **NP** \subseteq **PCP**[*poly*, 1]

Proof: It suffices to come up with a PCP system for one NP-complete language, where the verifier

- uses polynomially many random bits (exponentially long proofs)
- makes a constant number of queries to that proof

Plan:

- an NP-complete language: Quadeq
- Walsh-Hadamard encodings
- a PCP[poly, 1] system for Quadeq



All arithmetic today will be modulo 2, that is, over the field $\{0, 1\}$!

- 1 + 1 = 0
- $x^2 = x$
- x + y = x y

Quadeq

- satisfiable quadratic equations over {0, 1}
- *n* variables/*m* equations
- no purely linear terms
- NP-complete (exercise!)

Example (Running example)

$$xy + xz = 1$$

 $y^2 + yz + z^2 = 1$
 $x^2 + yx + z^2 = 0$

Quadeq

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Example (Running example)

$$xy + xz = 1$$

 $y^2 + yz + z^2 = 1$
 $x^2 + yx + z^2 = 0$

Solution: x = 1, y = 0, z = 1as a vector: **s** = (1 0 1)

Be smart, use vector notation

$$xy + xz = 1y2 + yz + z2 = 1x2 + yx + z2 = 0s = (1 0 1)$$

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vector notation: for a given $m \times n^2$ matrix A and m vector **b** find solution $\mathbf{u} = (x \ y \ z)$ such that

 $A(\mathbf{u}\otimes\mathbf{u})=\mathbf{b}$

u⊗u	x ²	хy	ХZ	yх	y ²	уz	ZX	zy	Z^2	
u⊗u s⊗s	1	0	1	0	0	0	1	0	1	b
Α	0	1	1	0	0	0	0	0	0	1
	0	0	0	0	1	1	0	0	1	1
	1	1 0 0	0	1	0	0	0	0	1	0

Overview

- Quadeq is the language of satisfiable systems of quadratic equations over {0, 1}
- natural PCP system expects a solution u and checks whether it is valid
- but this yields superconstant number of queries!
- how can we encode a solution such that a constant number of queries suffices?

Overview

- Quadeq is the language of satisfiable systems of quadratic equations over {0, 1}
- natural PCP system expects a solution u and checks whether it is valid
- but this yields superconstant number of queries!
- how can we encode a solution such that a constant number of queries suffices?
- use longer proofs!
- an NP-complete language: Quadeq √
- Walsh-Hadamard encodings
- a PCP[poly, 1] system for Quadeq

PCP for Quadeq

Input: $m \times n^2$ matrix A, m vector b					
Verifier	Proof π				
 check that <i>f</i>, <i>g</i> are linear functions check that <i>g</i> = WH(u ⊗ u) where <i>f</i> = WH(u) check that <i>g</i> encodes a satisfying assignment 	 π ∈ {0, 1}^{2ⁿ+2^{n²}} π is a pair of linear functions ⟨f, g⟩, i.e. strings from {0, 1}^{2ⁿ} and {0, 1}^{2^{n²}}, resp. if u satisfies A(u ⊗ u) = b then f = WH(u) and g = WH(u ⊗ u) are Walsh-Hadamard encodings 				

Walsh-Hadamard encoding

Definition (WH)

Let $\mathbf{u} \in \{0, 1\}^n$ be a vector. The Walsh-Hadamard encoding of \mathbf{u} written $WH(\mathbf{u})$ is the truth table of the linear function $f : \{0, 1\}^n \to \{0, 1\}$ with $f(\mathbf{x}) = \mathbf{u} \odot \mathbf{x}$ where $(u_1 \ldots u_n) \odot (x_1 \ldots x_n) = \sum_{i=1}^n u_i x_i$.

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Example

The solution to our running example is $s = (1 \ 0 \ 1)$. We have

 $WH(\mathbf{s}) = (0\ 1\ 0\ 1\ 1\ 0\ 1\ 0)$

Note: $|WH(\mathbf{u})| = 2^{|\mathbf{u}|}$

Properties (without proof)

Random subsum principle

- if $\mathbf{u} \neq \mathbf{v}$ then for 1/2 of the choices of \mathbf{x} we have $\mathbf{u} \odot \mathbf{x} \neq \mathbf{v} \odot \mathbf{x}$
- if $\mathbf{u} \neq \mathbf{v}$ then $WH(\mathbf{u})$ and $WH(\mathbf{v})$ differ on at least half their bits

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Local linearity testing

• we say that $f, g: \{0, 1\}^n \rightarrow \{0, 1\}$ are ρ -close if

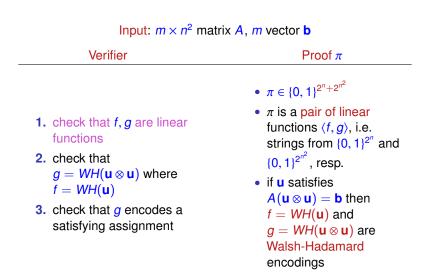
 $Pr_{\mathbf{x}\in_{R}\{0,1\}^{n}}[f(\mathbf{x})=g(\mathbf{x})]\geq\rho$

• if there exists a $\rho > 1/2$ s.t.

$$Pr_{\mathbf{x},\mathbf{y}\in_{R}\{0,1\}^{n}}[f(\mathbf{x}+\mathbf{y})=f(\mathbf{x})+f(\mathbf{y})]\geq
ho$$

then *f* is ρ -close to a linear function

PCP for Quadeq



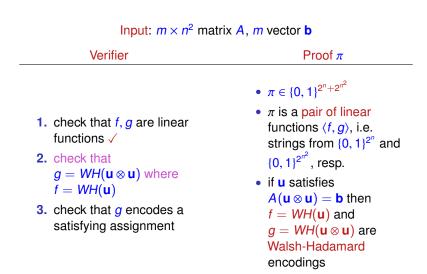
Local linearity testing

- we test the linearity condition (f(x + y) = f(x) + f(y)) independently $1/\delta > 2$ times, and accept if all tests pass
- we accept a linear function with probability 1
- if f is not 1δ -close to a linear function
 - all tests are passed with probability at most $(1 \delta)^{(1/\delta)}$
 - \Rightarrow such a function is rejected with probability at least 1 1/e > 1/2
- for instance, we could make a 0.999 linearity test using 1000 trials

Local decoding

- it might happen, that we accept non-linear functions that are very close to linear functions
- · in this case we treat them as if they were linear
- if we want to query f(x)
 - **1.** we choose $\mathbf{x}' \in \{0, 1\}^n$ at random
 - 2. set x'' = x + x'
 - **3.** let y' = f(x') and y'' = f(x'')
 - **4.** output **y**' + **y**''
- this makes two queries instead of one
- and recovers the value of the closest linear function with high probability

PCP for Quadeq



Check WH encodings

Test 10 times for random $\mathbf{r}, \mathbf{r}' \in \{0, 1\}^n$

 $f(\mathbf{r})f(\mathbf{r}') = g(\mathbf{r}\otimes\mathbf{r}')$

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If the proof is correct we always accept:

$$f(\mathbf{r})f(\mathbf{r}') = (\sum_{i \in [n]} u_i r_i) (\sum_{j \in [n]} u_j r_j')$$

$$= \sum_{i,j \in [n]} u_i u_j r_i r_j'$$

$$= ((\mathbf{u} \otimes \mathbf{u}) \odot (\mathbf{r} \otimes \mathbf{r}'))$$

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If the proof is wrong we reject with probability at least 1/4 by applying the random subsum principle twice, because in esence we compute rUr' and rWr' for different matrices U and W.

PCP for Quadeq

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Is the assignment satisfying?

- for each of *m* equations we can check g(z) at some place z corresponding to the coefficients in matrix A
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- for each of *m* equations we can check g(z) at some place z corresponding to the coefficients in matrix A
- but this is not constant queries!
- instead multiply each equation by a random bit and take the sum of all equations
- if g encodes a solution, we will always have a solution to the sum
- otherwise, we have a solution with probability 1/2 only

Is the system in PCP[poly(n), 1]?

1. $\pi \in \{0, 1\}^{2^n + 2^{n^2}}$

- 2. check that f, g are linear functions
 - $2(1-\delta) \cdot n$ random bits, $2(1-\delta)$ queries
- **3.** check that $g = WH(\mathbf{u} \otimes \mathbf{u})$ where $f = WH(\mathbf{u})$
 - 20n random bits, 20 queries
- 4. check that g encodes a satisfying assignment
 - *m* random bits (one per equation), 1 query

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Yes!

Conclusion

PCP and hardness of approximation

- computing approximate solutions to NP-hard problems is important
- the classical Cook-Levin reduction does not rule out efficient approximations
- many nontrivial approximation algorithms exist (2-app for metric TSP, knapsack, 2-app for vertex cover)
- PCP theorem shows hardness of approximating max3SAT to within any constant factor if P ≠ NP
- we showed hardness of approximation for Indset as well
- this is equivalent to having a probabilistically checkable proof system with logarithmic randomness and constant queries
- PCP proofs involve intricate encoding schemes like Walsh-Hadamard

Further Reading Luca Trevisan, Inapproximability of Combinatorial Optimization Problems, available from http://www.cs.berkeley.edu/~luca/pubs/inapprox.pdf Next and final topic: Parallelism