Complexity Theory

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Lecture 2

Turing Machines

Agenda

Formalize a model of computation!

- k-tape Turing machines
- robustness
- universal Turing machine
- computability, halting problem
- **P**

Which models of computation do you know?

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- programming languages
- hardware
- biological/chemical systems
- primitive/μ-recursive functions/λ-calculus
- logic
- automata
- quantum computers
- paper and pencil

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Turing machines!

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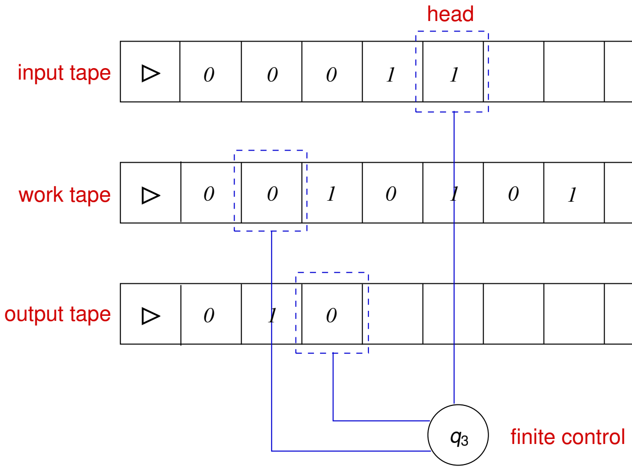
Turing machines!

Church-Turing Thesis: all models equally expressive

TMs – illustrated



k-tape Turing machines

- *k* scratchpad tapes, infinitely long, contain cells
  - one input tape, read-only
  - one output tape
  - working tapes
  - *k* heads positioned on individual cells for reading and writing
- finite control (finite set of rules)
- vocabulary, alphabet to write in cells
- actions: depending on
  - symbols under heads
  - control state

  one can
  - move heads (right, left, stay)
  - write symbols into current cells

TMs – reading palindromes

TM for function $pal : \{0, 1\}^* \to \{0, 1\}$ which outputs 1 for palindromes.

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TM for function $pal : \{0,1\}^* \to \{0,1\}$ which outputs 1 for palindromes.

- copy input to work tape
- move input head to front, work tape head to end
- in each step
  - compare input and work tape
  - move input head right
  - move work head left
- if whole input processed, output 1

TMs – formally

Definition (k-tape Turing machine (syntax))

Turing machine is a triple (Γ, Q, δ) where

- Γ is a finite alphabet (tape symbols) comprising 0, 1, $\square$ (empty cell), and $\triangleright$ (start symbol)
- Q is finite set of states (control) containing q_{start} and q_{halt}
- $\delta : Q \times \Gamma^k \to Q \times \Gamma^{k-1} \times \{l, s, r\}^k$, transition function such that $\delta(q_{halt}, \vec{\sigma}) = (q_{halt}, \vec{\sigma}_{2..k}, \vec{s})$.

TMs – formally

Definition (Computing a function and running time)

Let M be a k-tape TM and $x \in (\Gamma \setminus \{\Box, \triangleright\})^*$ an input. Let $T : \mathbb{N} \to \mathbb{N}$ and $f : \{0,1\}^* \to \{0,1\}^*$ be functions.

1. the start configuration of M on input x is $\triangleright x \Box^\omega$ on the input tape and $\triangleright \Box^\omega$ on the $k-1$ other tapes; all heads are on $\triangleright$; and M is in state q_{start}
2. if M is in state q and $(\sigma_1, \ldots, \sigma_k)$ are symbols being read, and $\delta(q, (\sigma_1, \ldots, \sigma_k)) = (q', (\sigma'_2, \ldots, \sigma'_k), \vec{z})$, then at the next step M is in state q', σ_i has been replaced by σ'_i for $i = 2..k$ and the heads have moved *l*eft, *s*tayed, or *r*ight according to $\vec{z}$
3. M has halted if it gets to state q_{halt}
4. M computes f in time T if it halts on input x with $f(x)$ on its output tape and every $x \in \{0,1\}^*$ requires at most $T(|x|)$ steps.

Remarks on TM definition

- TMs are deterministic
- going left from $\triangleright$ means staying
- item 4: consider time-constructible functions T only
  - $T(n) \geq n$ and
  - exists TM M computing T in time T
- TM define total functions

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Robustness

Definition of TM is robust, most choices do not change complexity classes.

- alphabet size (two is enough)
- number of tapes (one is enough)
- tape dimensions (one-directional tapes, bi-directional tapes, two-dimensional tapes)
- random access TMs
- oblivious TMs
  - see exercises
  - head positions at i-th step of execution on input x depend only on $|x|$ and i

All variations can simulate each other with at most polynomial overhead in running time.

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Universal TM

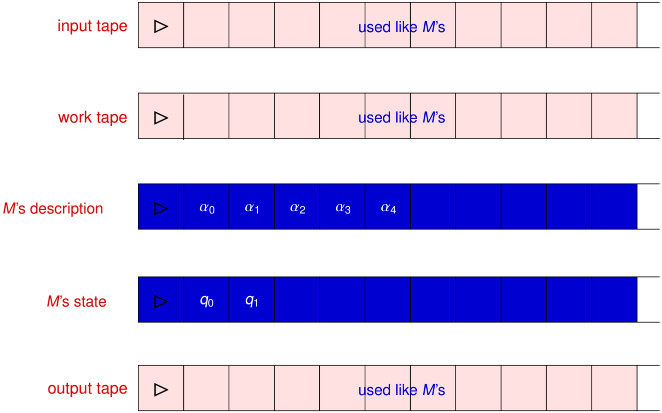
- TMs can be represented as strings (over $\{0, 1\}$) by encoding their transition function (can you?)
  - write M_α for TM represented by string α
  - every string α represents some TM
  - every TM has infinitely many representations
- if TM M computes f, universal TM $\mathcal{U}$ takes representation α of TM M and input x and computes $f(x)$
- like general purpose computer loaded with software
- like interpreter for a language written in same language
- $\mathcal{U}$ has bounded alphabet, rules, tapes; simulates much larger machines efficiently

Efficient simulation

Theorem (Universal TM)

There exists a TM $\mathcal{U}$ such that for every $x, \alpha \in \{0,1\}^*$, $\mathcal{U}(x, \alpha) = M_\alpha(x)$. If M_α holds on x within T steps, then $\mathcal{U}(x, \alpha)$ holds within $\mathcal{O}(T \log T)$ steps.

Construction of $\mathcal{U}$



Simulating another TM

How does $\mathcal{U}$ execute TM M?

Simulating another TM

How does $\mathcal{U}$ execute TM M?

1. transform M into M' with one input, one work, and one output tape computing the same function — quadratic overhead
2. write M''s description α onto third tape — $|M'|$
3. write encoding of M' start state on fourth tape — $|Q'|$
4. for each step of M'
   - 4.1 depending on state and tapes of M' scan δ' tape — $|\delta'|$
   - 4.2 update — constant

Simulation can be done with logarithmic slowdown using clever encoding of k tapes in one.

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Deciding languages

- often one is interested in functions $f : \{0,1\}^* \to \{0,1\}$
- f can be identified with the language $L_f = \{x \in \{0,1\}^* \mid f(x) = 1\}$
- TM that computes f is said to decide L_f (and vice versa)

Halting Problem

There are languages that cannot be decided by any TM regardless time and space.

Example

The halting problem is the set of pairs of TM representations and inputs, such that the TMs eventually halt on the given input.

$$\mathsf{Halt} = \{\langle \alpha, x \rangle \mid M_\alpha \text{ halts on } x\}$$

Theorem

Halt *is not decidable by any TM.*

Proof: diagonalization and reduction

DTIME

Definition (DTIME)

Let $T : \mathbb{N} \to \mathbb{N}$ be a function. $L \subseteq \{0,1\}^*$ is in **DTIME**(T) if there exists a TM deciding L in time T' for $T' \in \mathcal{O}(T)$.

- D refers to deterministic
- constants are ignored since TM can be sped up by arbitrary constants

P

Definition (P)

$$\mathbf{P} = \bigcup_{c \geq 1} \mathbf{DTIME}(n^c)$$

- **P** captures tractable computations
- low-level choices of TM definitions are immaterial to **P**
- Connectivity, Primes $\in$ **P**

What have we learnt?

- many equivalent ways to capture essence of computations (Church-Turing)
- k-tape TMs
- TM can be represented as strings; universal TM can simulate any TM given its representations with polynomial overhead only
- uncomputable functions do exist (halting problem): diagonalization and reductions
- **P** robust wrt. tweaks in TM definition (universal simulation)
- **P** captures tractable computations, solvable by TMs in polynomial time
- diagonalization, reduction
- up next: **NP**