Complexity Theory

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Lecture 14

Interactive Proofs

Overview

NP certificates or proof of membership

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NP certificates or proof of membership

$\downarrow$

RP proofs chosen at random

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NP certificates or proof of membership

↓

RP proofs chosen at random

↓

IP interactive proofs
between a prover and a verifier

Example: job interview, interactive vs. fixed questions

Agenda

- interactive proof examples
  - socks
  - graph coloring
  - graph non-isomorphism
- definition of interactive proof complexity
  - **IP**
  - public coins: **AM**

Different socks

Example

P wants to convince V that she has a red sock and a yellow sock.
V is blind and has a coin.

Interactive Proof

1. P tells V which sock is red
2. V holds red sock in her right hand, left sock in her yellow hand
3. P turns away from V
4. V tosses a coin
   4.1 heads: keep socks
   4.2 tails: switch socks
5. V asks P where the red sock is

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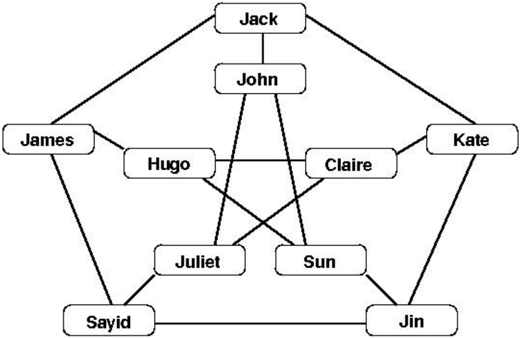
- random choices are crucial
- P has more computational power (vision) than V
- P must not see V's coin (private coin)

Graph 3-Coloring



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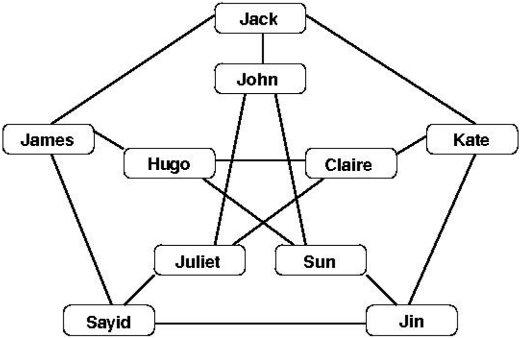
- How can she prove it to V?

Graph 3-Coloring



- P claims: G is 3-colorable
- How can she prove it to V?
- provide certificate (since $3\!-\!\mathrm{Col} \in$ **NP**), V checks it
- possible for all $L \in$ **NP** with one round if P has **NP** power

What if actual coloring should be secret?

- given a graph (V, E) with $|V| = n$
- P claims 3-colorability
- P wants to convince V of coloring $c : V \to C \quad (= \{R, G, B\})$

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- given a graph (V, E) with $|V| = n$
- P claims 3-colorability
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1. P randomly picks a permutation $\pi : C \to C$ and puts $\pi(c(v_i))$ in envelope i for each $1 \leq i \leq n$
2. V randomly picks edge (u_i, u_j) and opens envelopes i and j to find colors c_i and c_j
3. V accepts iff $c_i \neq c_j$

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$\Rightarrow$ zero-knowledge protocol

- by reductions, all **NP** languages have ZK protocols
- private coins

Graph Non-Isomorphism

- **NP** languages have succinct, deterministic proofs
- **coNP** languages possibly don't
- graph isomorphism, GI, is in **NP**
- hence GNI $= \{\langle G_1, G_2 \rangle \mid G_1 \not\cong G_2\}$ is in **coNP**
- GNI has a succinct interactive proof

Interactive Proof for GNI

given: graphs G_1, G_2

- **V** pick $i \in_R \{1, 2\}$, random permutation π
- **V** use π to permute nodes of G_i to obtain graph H
- **V** send H to P
- **P** check which of G_1, G_2 was used to obtain H
- **P** let G_j be that graph and send j to V
- **V** accept iff $i = j$

Intuition

- same idea as for socks protocol
- P has unlimited computational power
- if $G_1 \cong G_2$ then P answers correctly with probability at most $1/2$
- probability can be improved by sequential or parallel repetition
- if $G_1 \ncong G_2$ then P answers correctly with probability 1
- privacy of coins crucial

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  - socks ✓
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  - graph non-isomorphism ✓
- definition of interactive proof complexity
  - **IP**
  - public coins: **AM**

Interaction

Definition (Interaction)

Let $f, g : \{0,1\}^* \to \{0,1\}^*$ be functions and $k \geq 0$ an integer that may depend on the input size. A k-round interaction of f and g on input $x \in \{0,1\}^*$ is the sequence $\langle f, g\rangle(x)$ of strings $a_1, \ldots, a_k \in \{0,1\}^*$ defined by

$$\begin{array}{rcll} a_1 & = & f(x) & \\ a_2 & = & g(x, a_1) & \\ & \cdots & & \\ a_{2i+1} & = & f(x, a_1, \ldots, a_{2i}) & \text{for } 2i < k \\ a_{2i+2} & = & g(x, a_1, \ldots, a_{2i+1}) & \text{for } 2i+1 < k \end{array}$$

The output of f at the end of the interaction is defined by $out_f\langle f, g\rangle(x) = f(x, a_1, \ldots, a_k)$ and assumed to be in $\{0,1\}$.

This is a deterministic interaction, we need to add randomness.

Adding Randomness

Definition (IP)

For an integer $k \geq 1$ that may depend on the input size, a language L is in $\mathbf{IP}[k]$, if there is a probabilistic polynomial-time TM V that can have a k-round interaction with a function $P : \{0,1\}^* \rightarrow \{0,1\}^*$ such that

- Completeness
  $x \in L \implies \exists P.Pr[out_V\langle V, P\rangle(x) = 1] \geq 2/3$
- Soundness
  $x \notin L \implies \forall P.Pr[out_V\langle V, P\rangle(x) = 1] \leq 1/3$

We define $\mathbf{IP} = \bigcup_{c \geq 1} \mathbf{IP}[n^c]$.

- V has access to a random variable $r \in_R \{0,1\}^m$
- e.g. $a_1 = f(x, r)$ and $a_3 = f(x, a_1, r)$
- g cannot see r

$\Rightarrow$ $out_V\langle V, P\rangle(x)$ is a random variable where all probabilities are over the choice of r

Arthur-Merlin Protocols

Definition (AM)

- For every k the complexity class **AM**$[k]$ is defined as the subset of **IP**$[k]$ obtained when the verfier's messages are random bits only and also the only random bits used by V.
- **AM** $=$ **AM**$[2]$

Such an interactive proof is called an Arthur-Merlin proof or a public coin proof.

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Basic Properties

- **NP** $\subseteq$ **IP**
- for every polynomial $p(n)$ the acceptance bounds in the definition of **IP** can be changes to
  - $2^{-p(n)}$for soundness
  - $1 - 2^{-p(n)}$ for completeness
- the requirement for completeness can be changed to require probability 1 yielding perfect completeness
- perfect soundness collapses **IP** to **NP**

What have we learnt?

- **IP**$[k]$: languages that have k-round interactive proofs
- interaction and randomization possibly add power
  - randomization alone: **BPP** (possibly equals **P**)
  - deterministic interaction: **NP**

  ⇒ interactive proofs more succinct
- prover has unlimited computational power
- verifier is a **BPP** machine (poly-time with coins)
- coins can be private or public
- zero-knowledge protocols do exist for all **NP** languages
- soundness and completeness thresholds can be adapted

What's next?

- $\mathbf{AM}[2] = \mathbf{AM}[k]$ AM hierarchy collapses
- $\mathbf{AM}[k+2] = \mathbf{IP}[k]$ private coins don't help
- if graph isomorphism is **NP**-complete, the polynomial hierarchy collapses
- $\mathbf{IP} = \mathbf{PSPACE}$