Complexity Theory

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Based on slides by Michael Luttenberger

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Lecture 12–13

Randomization and Polynomial Time

"Realistic computation somewhere between P and NP"

Agenda

- Motivation: From **NP** to a more realistic class by randomization
  - Choosing the certificate at random
  - Error reduction by rerunning
- Randomized poly-time with one-sided error: **RP**, **coRP**, **ZPP**
- Power of randomization with two-sided error: **PP**, **BPP**

Recap P

Definition (P)

For every $L \subseteq \{0,1\}^*$:
$L \in \mathbf{P}$ if there is a poly-time TM M such that for every $x \in \{0,1\}^*$:

$$x \in L \Leftrightarrow M(x) = 1.$$

- "poly-time TM M":
  - M deterministic
  - M outputs $\{0,1\}$
  - There is a polynomial $T(n)$ s.t. M halts on every x within $T(|x|)$ steps.
- Problems in **P** are deemed "tractable".

Recap NP

Theorem (Certificates)

For every $L \subseteq \{0,1\}^*$:
*$L \in$ **NP** if and only if there exists a polynomial $p : \mathbb{N} \to \mathbb{N}$ and a poly-time TM M such that for every $x \in \{0,1\}^*$*

$$x \in L \Leftrightarrow \exists u \in \{0,1\}^{p(|x|)} \;:\; M(x,u) = 1$$

- Certificate u: satisfying assignment, independent set, 3-coloring, etc.
- **NP** captures the class of possibly (not) tractable computations:
  - Don't know how to compute u in poly-time, but
  - if there is a u, then $|u|$ is polynomial in $|x|$, and
  - we can check in poly-time if a u is a certificate/solution.

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- Seems unrealistic. Common conjecture: **P** $\neq$ **NP**.

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$L \in \mathbf{NP}$ *if and only if there exists a polynomial* $p : \mathbb{N} \to \mathbb{N}$ *and a poly-time TM* M *such that for every* $x \in \{0,1\}^*$

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- NDTMs can check all $2^{p(|x|)}$ possible us in parallel.
- Seems unrealistic. Common conjecture: $\mathbf{P} \neq \mathbf{NP}$.
- Goal: Obtain from **NP** a more realistic class by randomization:

Choose u uniformly at random from $\{0,1\}^{p(|x|)}$.

Randomizing NP

Definition (Accept/Reject certificates and probabilities)

Fix some $L \in \mathbf{NP}$ decided by M using certificates u of length $p(\cdot)$:

$$A_{M,x} := \{u \in \{0,1\}^{p(|x|)} \mid M(x,u) = 1\} \text{ and } R_{M,x} := \{0,1\}^{p(|x|)} \setminus A_{M,x}.$$

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- If we choose $u \in \{0,1\}^{p(|x|)}$ uniformly at random:
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Definition (Accept/Reject certificates and probabilities (cont'd))

$$\Pr[A_{M,x}] := \frac{|A_{M,x}|}{2^{p(|x|)}} \text{ and } \Pr[R_{M,x}] := \frac{|R_{M,x}|}{2^{p(|x|)}} = 1 - \Pr[A_{M,x}].$$

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$L \in \mathbf{NP}$ iff $\forall x \in \{0,1\}^*$:

$$x \in L \Rightarrow \Pr[A_{M,x}] \geq 2^{-p(|x|)} \text{ and } x \notin L \Rightarrow \Pr[A_{M,x}] = 0.$$

Randomizing NP: Example SAT

- Input: CNF-formula ϕ with n variables.
- Output: Choose truth assignment $u \in \{0, 1\}^n$ uniformly at random.
  - If u satisfies ϕ, output yes, $\phi \in$ SAT.
  - Else, output probably, $\phi \notin$ SAT.

- If output is yes, $\phi \in$ SAT, then we know $\phi \in$ SAT for sure.
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- Consider $\phi = x_1 \wedge x_2 \wedge \ldots \wedge x_n \in$ SAT:
  - Probability of probably, $\phi \notin$ SAT: $\Pr[R_{M,x}] = 1 - 2^{-n}$
  - Called false negative.

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- If we run this algorithm r-times, prob. of false negative decreases to: $(1 - 2^{-n})^r \approx e^{-r/2^n}$.
- Exponential number $r \sim 2^n$ required to reduce this to any tolerable error bound like $1/4$ or $1/10$.
- Not that helpful as SAT $\in$ **EXP** (zero prob. of false negative).

Randomizing NP: Conclusion

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  Polynomial number $r(|x|)$ of reruns should make prob. of false negatives arbitrary small.
- This holds if $\Pr[A_{M,x}] \geq n^{-k}$ for some $k > 0$:

$$(1 - \Pr[A_{M,x}])^{c|x|^{k+d}} \geq \left(1 - 1/|x|^k\right)^{c|x|^{k+d}} \approx e^{-c|x|^d}$$

as $\lim_{m\to\infty}(1 - 1/m)^m = e^{-1}$.

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  - Definitions
  - Monte Carlo and Las Vegas algorithms
  - Examples: ZEROP and perfect matchings
- Power of randomization with two-sided error: **PP**, **BPP**

Definition of RP

Definition (Randomized P (RP))

$L \in \mathbf{RP}$ if there exists a polynomial $p : \mathbb{N} \to \mathbb{N}$ and a polynomial-time TM $M(x, u)$ using certificates u of length $|u| = p(|x|)$ such that for every $x \in \{0, 1\}^*$

$$x \in L \Rightarrow \Pr[A_{M,x}] \geq 3/4 \text{ and } x \notin L \Rightarrow \Pr[A_{M,x}] = 0.$$

- $\mathbf{P} \subseteq \mathbf{RP} \subseteq \mathbf{NP}$
- $\mathbf{coRP} := \{\overline{L} \mid L \in \mathbf{RP}\}$
- **RP** unchanged if we replace $\geq 3/4$ by $\geq n^{-k}$ or $\geq 1 - 2^{-n^k}$ $(k > 0)$.

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- Realistic model of computation? How to obtain random bits?
  - “Slightly random sources”: see e.g. Papadimitriou p. 261
- One-sided error probabiliy for **RP**:
  - False negatives: if $x \in L$, then $\Pr[R_{M,x}] \leq 1/4$.
  - If $M(x, u) = 1$, output $x \in L$; else output probably, $x \notin L$
  - Error reduction by rerunning a polynomial number of times.

coRP, ZPP

Lemma (coRP)

*$L \in$ **coRP** if and only if there exists a polynomial $p : \mathbb{N} \to \mathbb{N}$ and a polynomial-time TM $M(x, u)$ using certificates u of length $|u| = p(|x|)$ such that for every $x \in \{0, 1\}^*$*

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Definition (“Zero Probability of Error”-P (ZPP))

$$\textbf{ZPP} := \textbf{RP} \cap \textbf{coRP}$$

- If $L \in$ **ZPP**, then we have both an **RP**- and a **coRP**-TM for L.

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RP-algorithms

- Assume $L \in$ **RP** decided by TM $M(\cdot, \cdot)$.
- Given input x:
  - Choose $u \in \{0, 1\}^{p(|x|)}$ uniformly at random.
  - Run $M(x, u)$.
  - If $M(x, u) = 1$, output: yes, $x \in L$.
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- If we rerun this algorithm exactly k-times:
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- Expected running time if we rerun till output yes, $x \in L$:
  - If $x \in L$:
    - Number of reruns geometrically distributed with success prob. $\geq 3/4$, i.e.,
    - the expected number of reruns is at most $4/3$.
    - Expected running time also polynomial.
  - If $x \notin L$:
    - We run forever.

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- Then we have Monte Carlo algorithms for both $x \in L$ and $x \in \overline{L}$.
- Given x:
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- More on expected running time vs. exact running time later on.

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ZEROP

- Given: Multivariate polynomial $p(x_1, \ldots, x_k)$, not necessarily expanded, but evaluable in polynomial time.
- Wanted: Decide if $p(x_1, \ldots, x_k)$ is the zero polynomial.

$$\left\| \begin{pmatrix} 0 & y^2 & xy \\ z & 0 & y \\ 0 & yz & xz \end{pmatrix} \right\| = -y^2(z \cdot xz - 0) + xy(z \cdot yz - 0) = -xy^2z^2 + xy^2z^2 = 0$$

- ZEROP $:=$ “All zero polynomials evaluable in polynomial time”.
- E.g. determinant: substitute values for variables, then use Gauß-elemination.
- Not known to be in **P**.

ZEROP

Lemma (cf. Papadimitriou p. 243)

Let $p(x_1, \ldots, x_k)$ be a nonzero polynomial with each variable x_i of degree at most d. Then for $M \in \mathbb{N}$:

$$\left|\{(x_1, \ldots, x_k) \in \{0, 1, \ldots, M-1\}^k \mid p(x_1, \ldots, x_k) = 0\}\right| \leq kdM^{k-1}.$$

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Let $X_1, \ldots, X_k$ be independent random variables, each uniformly distributed on $\{0, 1, \ldots, M-1\}$. Then for $M = 4kd$:

$$p \notin \text{ZEROP} \Rightarrow \Pr[p(X_1, \ldots, X_k) = 0] \leq \frac{kdM^{k-1}}{M^k} = \frac{kd}{M} = \frac{1}{4}.$$

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  - we can evaluate $p(\cdot)$ in polynomial time, and
  - d is polynomial in the representation of p.
- See Arora p. 130 for work around if d is exponential
  - E.g. $p(x) = (\ldots((x-1)^2)^2 \ldots)^2$.

Perfect Matchings in Bipartite Graphs

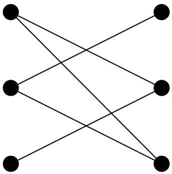
- Given: bipartite graph $G = (U, V, E)$ with

$$|U| = |V| = n \text{ and } E \subseteq U \times V$$

- Wanted: $M \subseteq E$ such that

$$\forall (u, v), (u', v') \in M : u \neq u' \wedge v \neq v' \quad \text{(matching)}$$

$$|M| = n \quad \text{(perfect)}$$



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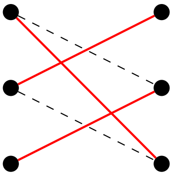

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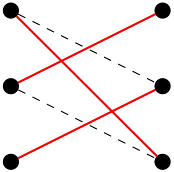


- Problem is known to be solvable in time $\mathcal{O}(n^5)$ (and better).
- So it is in **RP**.

Perfect Matchings in Bipartite Graphs

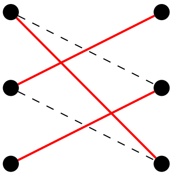
- Given: bipartite graph $G = (U, V, E)$ with

$$|U| = |V| = n \text{ and } E \subseteq U \times V$$

- Wanted: $M \subseteq E$ such that

$$\forall (u, v), (u', v') \in M : u \neq u' \land v \neq v' \quad \text{(matching)}$$

$$|M| = n \quad \text{(perfect)}$$



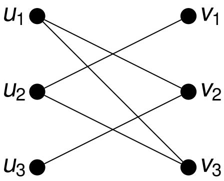
- Problem is known to be solvable in time $\mathcal{O}(n^5)$ (and better).
- So it is in **RP**.
- Still, some “easy” randomized algorithm relying on ZEROP.

Perfect Matchings in Bipartite Graphs

- For bipartite graph $G = (U, V, E)$ define square matrix M:

$$M_{ij} = \begin{cases} x_{ij} & \text{if } (u_i, v_j) \in E \\ 0 & \text{else }. \end{cases}$$

- Output:
  - “has perfect matching” if $\mathtt{det}(M) \notin \mathrm{ZEROP}$
  - “might not have perfect matching” if $\mathtt{det}(M) \in \mathrm{ZEROP}$



$$\left|\begin{pmatrix} 0 & x_{1,2} & x_{1,3} \\ x_{2,1} & 0 & x_{2,3} \\ 0 & x_{3,2} & 0 \end{pmatrix}\right| = x_{1,3}x_{2,1}x_{3,2}$$

- Relies on Leibniz formula: $\mathtt{det}\, M = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} M_{i,\sigma(i)}$.

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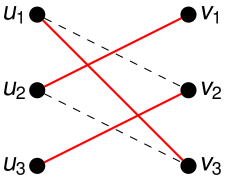

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Agenda

- Motivation: From **NP** to a more realistic class by randomization ✓
- Randomized poly-time with one-sided error: **RP**, **coRP**, **ZPP** ✓
  - Definitions ✓
  - Monte Carlo and Las Vegas algorithms ✓
  - Examples: ZEROP and perfect matchings ✓
- Power of randomization with two-sided error: **PP**, **BPP**
  - Enlarging **RP** by false negatives and false positives
  - Comparison: **NP**, **RP**, **coRP**, **ZPP**, **BPP**, **PP**
  - Probabilistic Turing machines
  - Expected running time
  - Error reduction for **BPP**
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  - **BPP** in the polynomial hierarchy

Probability of error for both $x \in L$ and $x \notin L$

- **RP** obtained from **NP** by
  - choosing certificate u uniformly at random
  - requiring a fixed fraction of accept-certificates if $x \in L$

$$x \in L \Rightarrow \Pr\left[A_{M,x}\right] \geq 3/4 \text{ and } x \notin L \Rightarrow \Pr\left[A_{M,x}\right] = 0.$$

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- Assume we change the definition of **RP** to:

$$x \in L \Leftrightarrow \Pr[A_{M,x}] \geq 3/4.$$

- Two-sided error probabilities:
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Probabilistic Polynomial Time (PP)

Definition (PP)

$L \in$ **PP** if there exists a polynomial $p : \mathbb{N} \to \mathbb{N}$ and a polynomial-time TM $M(x, u)$ using certificates u of length $|u| = p(|x|)$ such that for every $x \in \{0, 1\}^*$

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- **PP**: "$x \in L$ iff x is accepted by a majority"
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- Next: **PP** is at least as untractable as **NP**.

$\mathbf{NP} \subseteq \mathbf{PP}$

Theorem

$$\mathbf{NP} \subseteq \mathbf{PP}$$

- Assume TM $M(x, u)$ for $L \in \mathbf{NP}$ uses certificates u of length $p(|x|)$.
- Consider TM $N(x, w)$ with $|w| = p(|x|) + 2$:
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- Possible fix:
  - Require bounds on both error probabilities.
  - "Bounded error probability of error"-**P**

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- **BPP** = "most comprehensive, yet plausible notion of realistic computation" (Papadimitriou p. 259)

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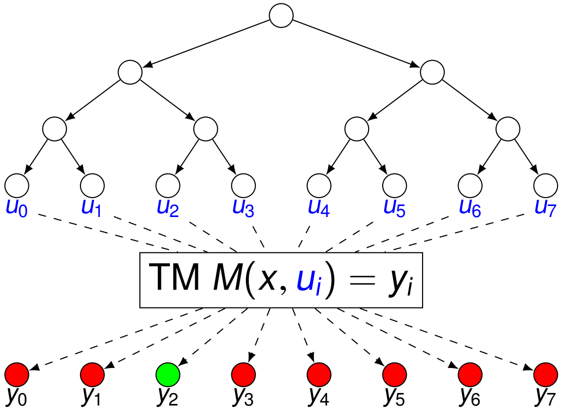
NP vs. RP vs. coRP vs. ZPP vs. BPP vs. PP



- $L \in$ **NP**:
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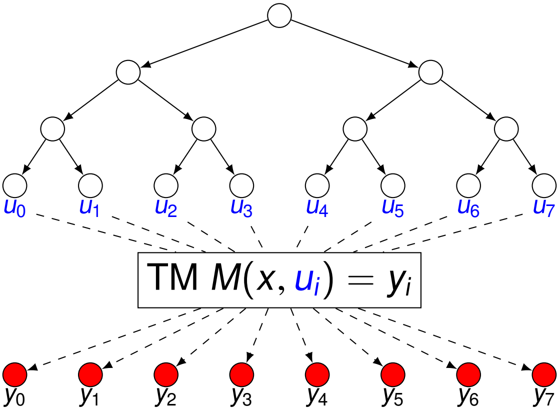

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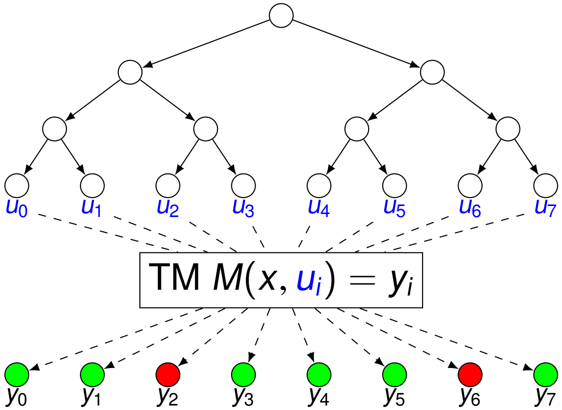

- $L \in$ **RP**:
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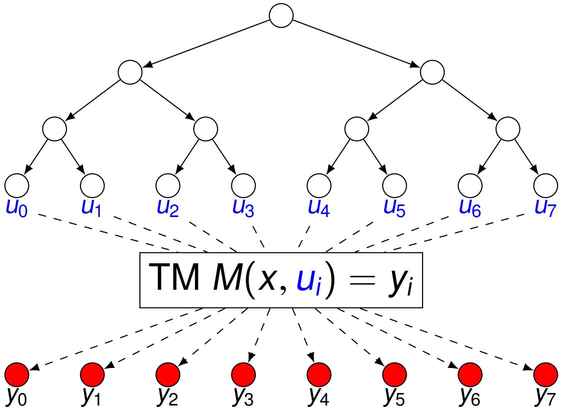

- $L \in$ **RP**:
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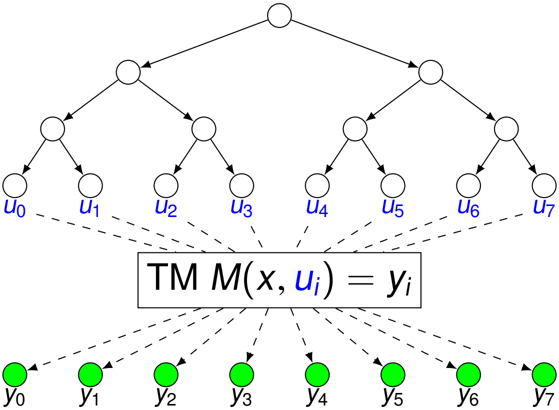

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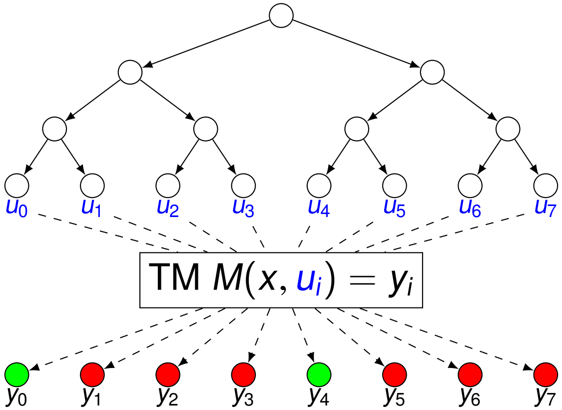

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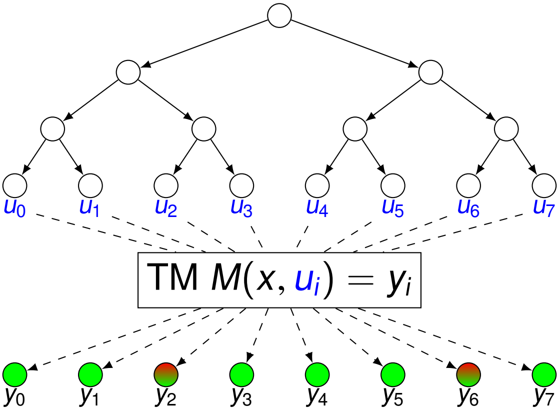

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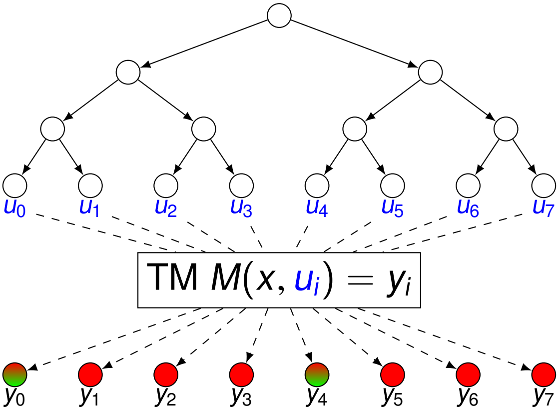

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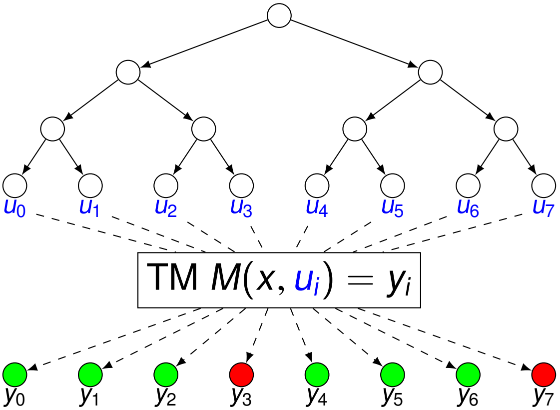

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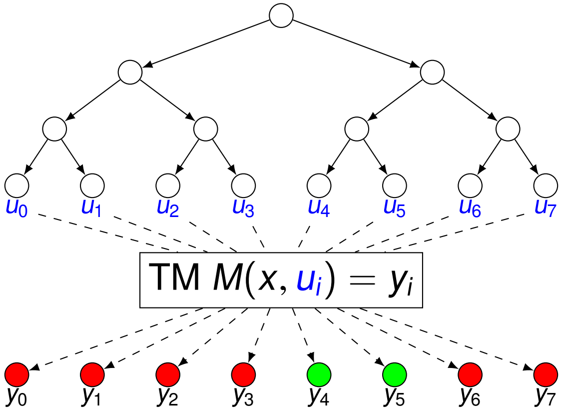

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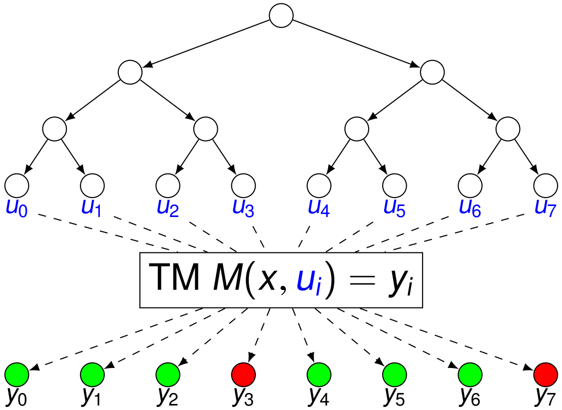

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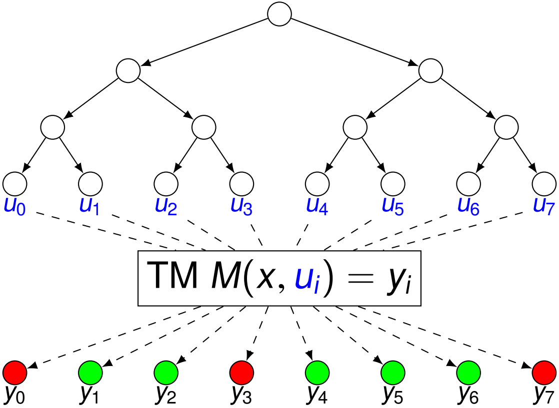

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Corollary

*$L \in$ **RP** iff there is a poly-time PTM M s.t. for all $x \in \{0,1\}^*$:*

$$x \in L \Rightarrow Pr[M(x) = 1] \geq 3/4 \text{ and } x \notin L \Rightarrow Pr[M(x) = 1] = 0.$$

Probabilistic Turing Machines

Definition (PTM)

We obtain from an NDTM $M = (\Gamma, Q, \delta_1, \delta_2)$ a probabilistic TM (PTM) by choosing in every computation step the transition function uniformly at random, i.e., any given run of M on x of length exactly l occurs with probability 2^{-l}.
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Corollary

*$L \in$ **BPP** iff there is a poly-time PTM M s.t. for all $x \in \{0,1\}^*$:*

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Agenda

- Motivation: From **NP** to a more realistic class by randomization ✓
- Randomized poly-time with one-sided error: **RP**, **coRP**, **ZPP** ✓
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Expected vs. Exact Running Time

- Recall: if $L \in$ **ZPP**
  - **RP**-algorithms for L and $\overline{L}$.
  - Rerun both algorithms on x until one outputs yes.
  - This decides L in expected polynomial time.
  - But might run infinitely long in the worst case.
- So, is expected time more powerful than exact time?

Expected Running Time

Definition (Expected running time of a PTM)

For a PTM M let $T_{M,x}$ be the random variable that counts the steps of a computation of M on x, i.e., $\Pr[T_{M,x} \leq t]$ is the probability that M halts on x within at most t steps.
We say that M runs in expected time $T(n)$ if $\mathbb{E}[T_{M,x}] \leq T(|x|)$ for every x.

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Definition (BPeP)

A language L is in **BPeP** if there is a polynomial $T : \mathbb{N} \to \mathbb{N}$ and a PTM M such that for every $x \in \{0,1\}^*$:

$$x \in L \Rightarrow \Pr[M(x) = 1] \geq 3/4 \text{ and } x \notin L \Rightarrow \Pr[M(x) = 0] \geq 3/4$$

and $\mathbb{E}[T_{M,x}] \leq T(|x|)$.

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- Assume $L \in$ **BPeP**.
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- Probability that M does more than k steps on input x:

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by Markov's inequality.

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  by Markov's inequality.
- So, for $k = 10T(|x|)$ (polynomial in $|x|$):

$$\Pr\left[T_{M,x} \geq 10T(|x|)\right] \leq 0.1$$

  for every input x.

Expected Running Time

- New algorithm $\tilde{M}$:
  - Simulate M for at most $10T(|x|)$ steps.
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  - In total: $1/4 \cdot \underbrace{\Pr[T_{M,x} \leq 10T(|x|)]}_{\leq 1} + 1/2 \cdot \underbrace{(1 - \Pr[T_{M,x} \leq 10T(|x|)])}_{\leq 0.1} \leq 0.3$

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Lemma

$$\mathbf{BPP} = \mathbf{BPeP}$$

Lemma

*$L \in$ **ZPP** iff L is decided by some PTM in expected polynomial time.*

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Error reduction

- Consider: $L \in$ **RP**:
  - Probability for error after r reruns:
  - if $x \notin L$: $= 0$
  - if $x \in L$: $\leq 4^{-r}$, i.e., r-times probably, $x \notin L$.

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- Similarly for $L \in$ **coRP** and $L \in$ **ZPP**.
- What if $L \in$ **BPP**?
  - We cannot wait for a yes
  - Instead use the majority.

Error reduction for BPP

Definition (BPP(f))

Let $f : \mathbb{N} \to \mathbb{Q}$ be a function.
$L \in \mathbf{BPP}(f)$ if there exists a polynomial $p : \mathbb{N} \to \mathbb{N}$ and a polynomial-time TM M such that for every $x \in \{0,1\}^*$

$$x \in L \Rightarrow \Pr\left[A_{M,x}\right] \geq f(|x|) \text{ and } x \notin L \Rightarrow \Pr\left[R_{M,x}\right] \geq f(|x|).$$

Theorem (Error reduction for BPP)

For any $c > 0$:

$$\mathbf{BPP} = \mathbf{BPP}(1/2 + n^{-c})$$

- The longer the input, the less dominant the “majority” has to be.

Error reduction for BPP (Proof)

- Assume $L \in$ **BPP**, and
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- So: $L \cap \{0,1\}^{\geq n_0} \in$ **BPP**$(1/2 + n^{-c})$.
- Thus, **BPP**$(1/2 + n^{-c})$-algorithm for L:
  - If $|x| < n_0$, decide $x \in L$ in **P** (error prob. $= 0$)
  - Else run **BPP**-algorithm (error prob. $\leq 1/4$)

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  - with $y_i \in \{0, 1\}$ and $y_i = 1$ if output probably, $x \in L$
  - $Y_1 = \sum_{i=1}^{r} y_i$ number of 1s
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$$x \in L \ : \ \Pr[y_i = 1] \geq 1/2 + |x|^{-c} \text{ resp. } x \notin L \ : \ \Pr[y_i = 0] \geq 1/2 + |x|^{-c}$$

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- Intuitively, for e.g. $r = |x|^{c+d}$ for some $d \in \mathbb{N}$ we get:

$$x \in L \ : \ \mathbb{E}[Y_1 - Y_0] \geq 2|x|^{d} \text{ resp. } x \notin L \ : \ \mathbb{E}[Y_0 - Y_1] \geq 2|x|^{d}$$

i.e., expect significant majority in favor of correct answer.

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- Thus:

$$\Pr[Y_1 \leq r/2] = \Pr[Y_1 \leq (1 - (1 - r/(2\mu)))\mu] \leq e^{-\mu\delta^2/2}$$

as long as $\delta := 1 - r/(2\mu) \in (0, 1)$.

Error reduction for BPP (Proof)

- Bounds on $\delta = 1 - r/(2\mu)$:

$$0 < \delta < 1 \Leftrightarrow 0 < r/2 < \mu \Leftarrow r/2 + r|x|^{-c} \leq \mu$$

- Thus, choose r s.t.

$$\Pr[Y_1 \leq r/2] \leq e^{-\mu\delta^2/2} \leq 1/4.$$

i.e.,

$$\mu\delta^2 \geq 2\log_e 4.$$

- With

$$\mu \geq r/2 + r|x|^{-c}$$

we obtain:

$$\mu\delta^2 = (\mu - r/2)(1-(r/2)/\mu)^2 \geq r|x|^{-c}\left(1 - \frac{r/2}{r/2 + r|x|^{-c}}\right)^2 = r \cdot \frac{|x|^{-3c}}{(1/2 + |x|^{-c})^2}$$

- So, choose $r \geq (\log_e 4) \cdot (|x|^{3c}/2 + 2|x|^{2c} + 2|x|^{c})$.

Error reduction for BPP (Proof)

- For $x \notin L$ we obtain analogously:

$$\Pr[Y_0 \leq Y_1] \leq 1/4 \text{ if } r \geq (|x|^{3c}/2 + 2|x|^{2c} + 2|x|^c).$$

- So, a polynomial number of rounds suffices to reduce error probability to at most $1/4$.
- Proof also yields:

Theorem (Error reduction for BPP)

For any $d > 0$:

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- Ex.: Show the theorem.

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Some Kind of Derandomization

Theorem

Let $L \in \mathbf{BPP}$ *be decided by a poly-time TM* $M(x, u)$ *using certificates of poly-length* $p(n)$*.*
Then for every $n \in \mathbb{N}$ *there exists a certificate* u_n *s.t. for all* x *with* $|x| = n$*:*

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- $\Pr\left[\bigcup_{|x|=n} B_x\right] \leq$

Some Kind of Derandomization

Theorem

Let $L \in \mathbf{BPP}$ be decided by a poly-time TM $M(x, u)$ using certificates of poly-length $p(n)$.
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- Seems unlikely for **NP**.

Agenda

- Motivation: From **NP** to a more realistic class by randomization ✓
- Randomized poly-time with one-sided error: **RP**, **coRP**, **ZPP** ✓
- Power of randomization with two-sided error: **PP**, **BPP**
  - Enlarging **RP** by false negatives and false positives ✓
  - Comparison: **NP**, **RP**, **coRP**, **ZPP**, **BPP**, **PP**✓
  - Probabilistic Turing machines✓
  - Expected running time✓
  - Error reduction for **BPP**✓
  - Some kind of derandomization for **BPP**✓
  - **BPP** in the polynomial hierarchy

BPP in the Polynomial Hierarchy PH

Theorem

$$\mathbf{BPP} \subseteq \mathbf{\Sigma_2^p} \cap \mathbf{\Pi_2^p}$$

- Reminder:
  - Definition of $L \in \mathbf{\Sigma_2^p}$:

$$x \in L \text{ iff } \exists u \in \{0,1\}^{p(|x|)} \forall v \in \{0,1\}^{p(|x|)} : M(x,u,v) = 1.$$

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$$x \in L \text{ iff } \forall u \in \{0,1\}^{p(|x|)} \exists v \in \{0,1\}^{p(|x|)} : M(x,u,v) = 1.$$

- As $\mathbf{BPP} = \mathbf{coBPP}$ it suffices to show $\mathbf{BPP} \subseteq \mathbf{\Sigma}_2^p$:

$$L \in \mathbf{BPP} \Rightarrow \overline{L} \in \mathbf{BPP} \Rightarrow \overline{L} \in \mathbf{\Sigma}_2^p \Rightarrow L \in \mathbf{\Pi}_2^p$$

BPP in the Polynomial Hierarchy PH

- We use again that **BPP** $=$ **BPP**$(1 - 4^{-n})$.
- Let $p(\cdot)$ be the polynomial bounding the certificate length.
- Recall $A_{M,x}$: "accept-certificates"

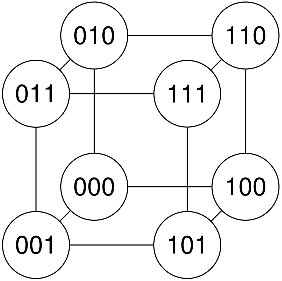
$$A_{M,x} := \{u \in \{0,1\}^{p(|x|)} \mid M(x,u) = 1\}$$

- Then

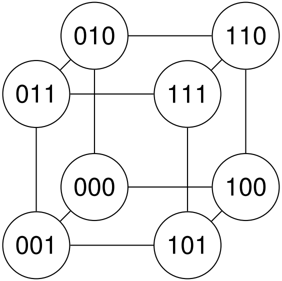
$$x \in L \Rightarrow |A_{M,x}| \geq (1 - 4^{-|x|})2^{p(|x|)} \text{ and } x \notin L \Rightarrow |A_{M,x}| \leq 4^{-n} \cdot 2^{p(|x|)}$$

- Need a formula to distinguish the two cases.

BPP in the Polynomial Hierarchy PH

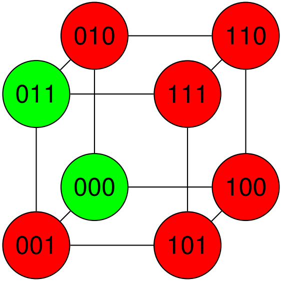


BPP in the Polynomial Hierarchy PH



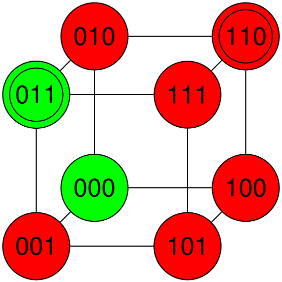
- Assume $|x| = 1$ and $p(|x|) = 3$,
- i.e., possible certificates in $\{0,1\}^3$.
- If $x \in L$, then $|A_{M,x}| \geq 3/4 \cdot 2^3 = 6$.
- If $x \notin L$, then $|A_{M,x}| \leq 1/4 \cdot 2^3 = 2$.

BPP in the Polynomial Hierarchy PH



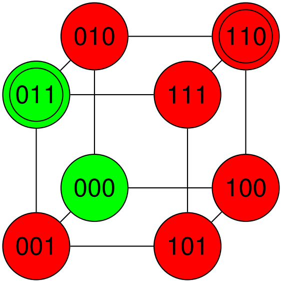
- Assume $x \notin L$, i.e., $|A_{M,x}| \leq 1/4 \cdot 8 = 2$

BPP in the Polynomial Hierarchy PH



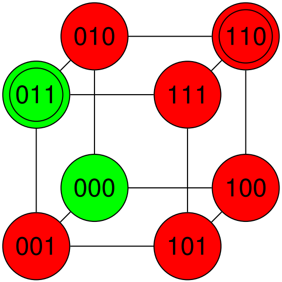
- Assume $x \notin L$, i.e., $|A_{M,x}| \leq 1/4 \cdot 8 = 2$
- Choose any $u_1, u_2 \in \{0,1\}^3$.

BPP in the Polynomial Hierarchy PH



- Assume $x \notin L$, i.e., $|A_{M,x}| \leq 1/4 \cdot 8 = 2$
- Choose any $u_1, u_2 \in \{0,1\}^3$.
- By chance, we might hit $A_{M,x}$.

BPP in the Polynomial Hierarchy PH

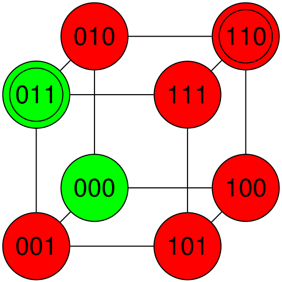


- Assume $x \notin L$, i.e., $|A_{M,x}| \leq 1/4 \cdot 8 = 2$
- Choose any $u_1, u_2 \in \{0,1\}^3$.
- By chance, we might hit $A_{M,x}$.
- Claim: But there is some $r \in \{0,1\}^3$ s.t.

$$\{u_1 \oplus r, u_2 \oplus r\} \cap A_{M,x} = \emptyset.$$

($\oplus$: bitwise xor)

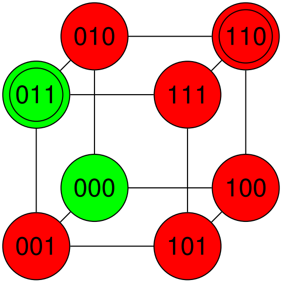
BPP in the Polynomial Hierarchy PH



- Note:

$$u_i \oplus r \in A_{M,x} \text{ iff } r \in A_{M,x} \oplus u_i.$$

BPP in the Polynomial Hierarchy PH



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- So, choose

$$r \in \overline{A_{M,x} \oplus u_1 \cup A_{M,x} \oplus u_2} = \overline{\{000, 011\} \cup \{101, 110\}}.$$

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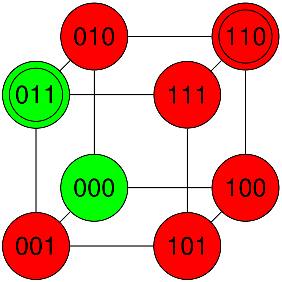

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- E.g. $r = 001$.

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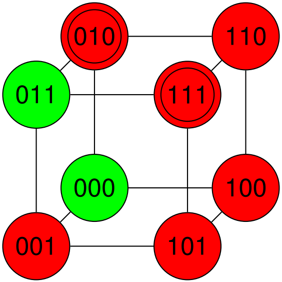

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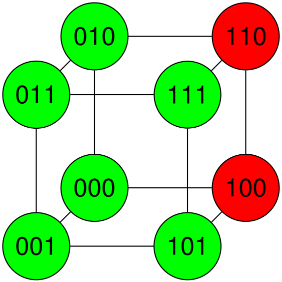
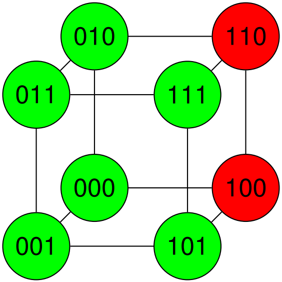


- Assume $x \in L$, i.e., $|A_{M,x}| \geq 6$.

BPP in the Polynomial Hierarchy PH

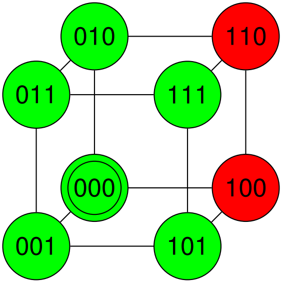


- Assume $x \in L$, i.e., $|A_{M,x}| \geq 6$.
- Claim: We can choose u_1, u_2 s.t. for any $r \in \{0,1\}^3$

$$\{u_1 \oplus r, u_2 \oplus r\} \cap A_{M,x} \neq \emptyset.$$

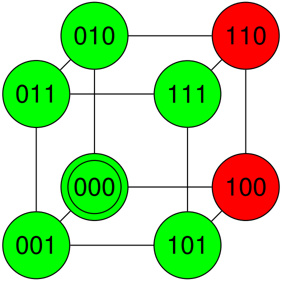
- Note: this is exactly the negation of the previous claim.

BPP in the Polynomial Hierarchy PH



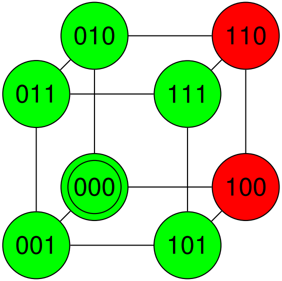
- E.g., take $u_1 = 000$.

BPP in the Polynomial Hierarchy PH



- E.g., take $u_1 = 000$.
- Then $u_1 \oplus r \in R_{M,x}$ iff $r \in u_1 \oplus R_{M,x} = \{100, 110\}$.

BPP in the Polynomial Hierarchy PH



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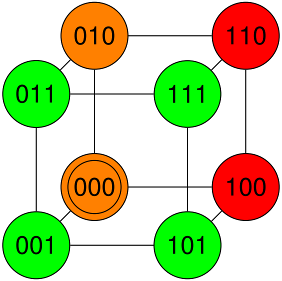

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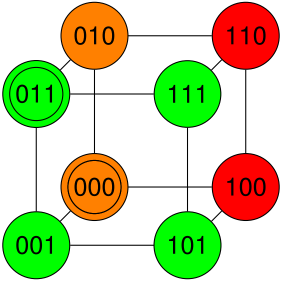

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- So, take $u_2 \notin 100 \oplus R_{M,x} \cup 110 \oplus R_{M,x}$.
- E.g., $u_2 = 011$.

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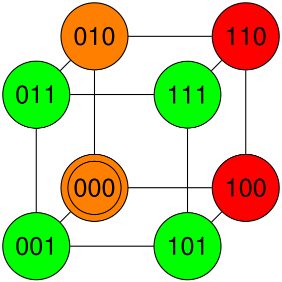

- Summary:

$$x \in L \cap \{0,1\}^1 \text{ iff } \exists u_1, u_2 \in \{0,1\}^3 \forall r \in \{0,1\}^3 \ : \bigvee_{i=1,2} u_i \oplus r \in A_{M,x}.$$

Reminder: $u_i \oplus r \in A_{M,x}$ iff $M(x, u_i \oplus r) = 1$.

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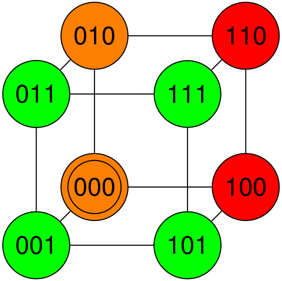

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  Reminder: $u_i \oplus r \in A_{M,x}$ iff $M(x, u_i \oplus r) = 1$.
- So, this is in $\mathbf{\Sigma_2^p}$.
- And works also for $|x| > 1$ and arbitrary $p(|x|)$.

BPP in the Polynomial Hierarchy PH

Claim:

Given x set $k := \lceil p(|x|)/|x| \rceil + 1$. Then:

$$x \in L \text{ iff } \exists u_1, \ldots, u_k \in \{0,1\}^{p(|x|)} \forall r \in \{0,1\}^{p(|x|)} : \bigvee_{i=1}^{k} M(x, u_i \oplus r) = 1.$$

- Note, the certificate $u_1 u_2 \ldots u_k$ has length polynomial in $|x|$.
- So, this formula represents a computation in $\mathbf{\Sigma_2^p}$.

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$$\bigwedge_{i=1}^{k} r \oplus u_i \notin A_{M,x} \equiv r \notin \bigcup_{i=1}^{k} u_i \oplus A_{M,x}.$$

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- Size of the complement of this set:

$$\left| \bigcup_{i=1}^{k} u_i \oplus A_{M,x} \right| \leq \sum_{i=1}^{k} \left| u_i \oplus A_{M,x} \right| = k \left| A_{M,x} \right| \leq k 4^{-|x|} 2^{p(|x|)} < 2^{p(|x|)}.$$

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- So, this set cannot be empty no matter how we choose $u_1, \ldots, u_k$.

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- Assume $x \in L$: To show there are $u_1, \ldots, u_k$ s.t.

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- For any given r: $\Pr\left[U_i \in r \oplus R_{M,x}\right] = \frac{|r \oplus R_{M,x}|}{2^{p(|x|)}} = \frac{|R_{M,x}|}{2^{p(|x|)}} \leq 4^{-n}$.
- $\Pr\left[\bigwedge_{i=1}^{k} U_i \in r \oplus R_{M,x}\right] = \Pr\left[U_1 \in r \oplus R_{M,x}\right]^k \leq 4^{-kn}$.

BPP in the Polynomial Hierarchy PH

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Given x set $k := \lceil p(|x|)/|x| \rceil + 1$. Then:

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- Assume $x \in L$: To show there are $u_1, \ldots, u_k$ s.t.

$$\forall r : \bigvee_{i=1}^{k} u_i \oplus r \in A_{M,x} \equiv \neg \exists r : \bigwedge_{i=1}^{k} u_i \in r \oplus R_{M,x}.$$

- Let $U_1, \ldots, U_k$ be independent random variables, uniformly distributed on $\{0,1\}^{p(|x|)}$. (Doesn't matter if some coincide!)
- For any given r: $\Pr[U_i \in r \oplus R_{M,x}] = \frac{|r \oplus R_{M,x}|}{2^{p(|x|)}} = \frac{|R_{M,x}|}{2^{p(|x|)}} \leq 4^{-n}$.
- $\Pr\left[\bigwedge_{i=1}^{k} U_i \in r \oplus R_{M,x}\right] = \Pr[U_1 \in r \oplus R_{M,x}]^k \leq 4^{-kn}$.
- $\Pr\left[\exists r : \bigwedge_{i=1}^{k} U_i \in r \oplus R_{M,x}\right] \leq \sum_{r \in \{0,1\}^*} 4^{-kn} = 2^{p(|x|)-2kn} < 1$.

BPP in the Polynomial Hierarchy **PH**

Claim:

Given x set $k := \lceil p(|x|)/|x| \rceil + 1$. Then:

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- For both cases there is an n_0 s.t. the bounds hold for all x with $|x| > n_0$.
- $L \cap \{0,1\}^{\leq n_0}$ can be decided trivially in **P**.

Summary

- Obtain **RP** from **NP** by
  - choosing the certificate (transition function) uniformaly at random
  - requiring a bound on $\Pr[A_{M,x}]$ if $x \in L$ s.t.
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- **ZPP** $:=$ **RP** $\cap$ **coRP** can be decided in expected polynomial time
  - Zero probability of error (if we wait for the definitive answer)
  - Las Vegas algorithms

Summary

- Obtained **PP** from **RP** by
  - allowing also for false positives
  - Error probabilities depend on each other: $\leq 1/4$ and $< 1 - 1/4$
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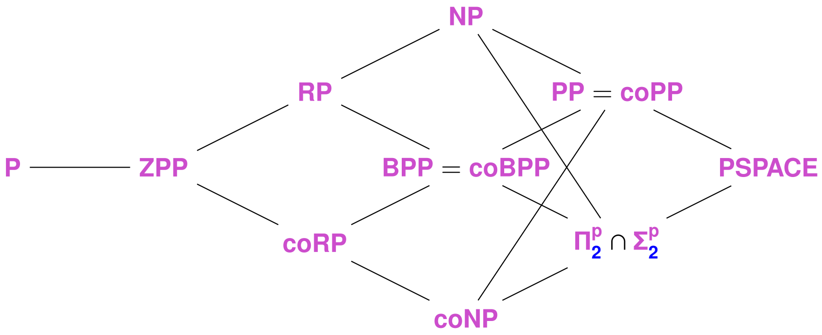
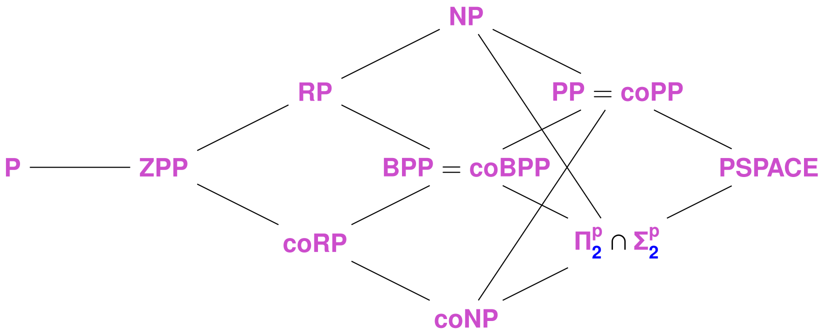
  - **NP** $\subseteq$ **PP**: "**PP** allows for trading one error prob. for the other"
- Obtained **BPP** from **PP** by
  - bounding both error prob. independently of each other.
  - Papadimitriou: "most comprehensive, yet plausible notion of realistic computation"
  - Conjecture: **BPP** $=$ **P**
  - Expected running time as powerful as exact running time.
  - One certificate u_n for all x with $|x| = n$.
  - Error reduction to 2^{-n^k} within a polynomial number of reruns.

Summary



- $\Pi_2^p \cap \Sigma_2^p \subseteq$ PP unknown.
- NP $\cup$ coNP $\subseteq$ PP known.

Summary



- Gödel Prize (1998) for Toda's theorem (1989): $\mathbf{PH} \subseteq \mathbf{P}^{\mathbf{PP}}$
  - $\mathbf{P}^{\mathbf{PP}}$: poly-time TMs having access to a **PP**-oracle.
  - If $\mathbf{PP} \subseteq \Sigma_k^p$ for some k, then $\mathbf{PH} = \Sigma_k^p$.
  - If $\mathbf{PP} \subseteq \mathbf{PH}$, then **PH** collapses at some finite level as **PP** has complete problems (see exercises).

Syntactic and Semantic Complexity Classes

- Just mentioned: **PP** has complete probems
  - $\phi \in$ MAJSAT iff at least $2^{n-1} + 1$ satisfying assignments of 2^n possible (see exercises).
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- Example:
  - **NP**:

    $$x \in L \Leftrightarrow \Pr[A_{M,x}] > 0.$$

    Every poly-time TM $M(x, u)$ defines a language in **NP**.
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    $$x \in L \Rightarrow \Pr[A_{M,x}] \geq 3/4 \text{ and } x \notin L \Rightarrow \Pr[R_{M,x}] \geq 3/4.$$

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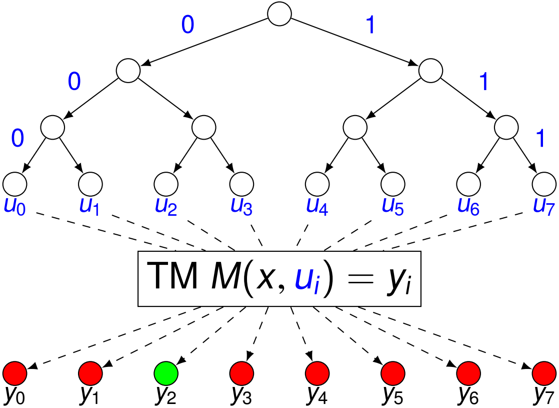
- Ex.: What about **PP**?

Leaf languages



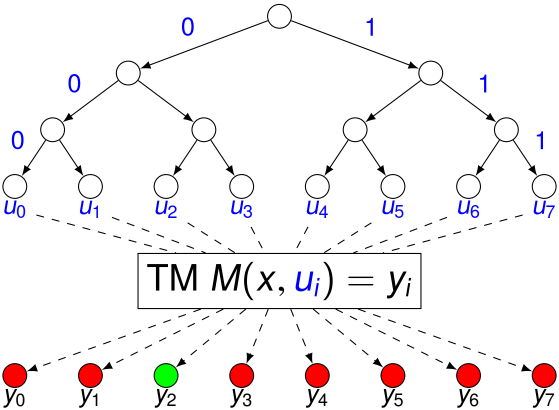
Definition

For a poly-time $M(x, u)$ using certificates $u \in \{0, 1\}^{p(|x|)}$ set

$$L_M(x) := y_0 y_1 \dots y_{2^{p(|x|)}-1} \text{ with } y_i = M(x, u_i) \text{ and } (u_i)_2 = i$$

The leaf-language of M is then $L_M := \{L_M(x) \mid x \in \{0, 1\}^*\}$.

Leaf languages

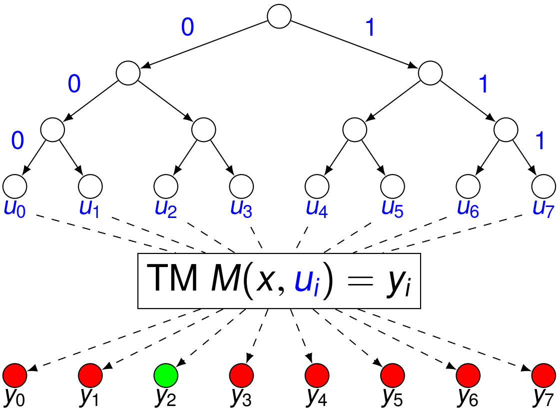


Definition (cont'd)

For $A, R \subseteq \{0,1\}^*$ with $A \cap R = \emptyset$ the class $\mathbf{C}[A, R]$ consists of all language L for which there is a TM $M(x, u)$ s.t. $\forall x \in \{0,1\}^*$:

$$x \in L \Rightarrow L_M(x) \in A \text{ and } x \notin L \Rightarrow L_M(x) \in R.$$

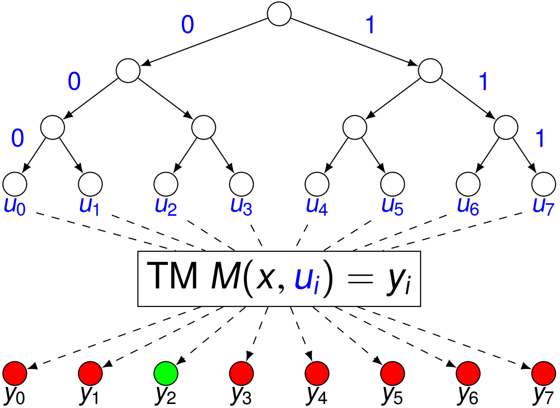
Leaf languages



Definition (cont'd)

C$[A, R]$ is called syntactic if $A \cup R = \{0, 1\}^*$, otherwise it is called semantic.

Leaf languages



- **NP** $=$ **C**$[(0+1)^*1(0+1)^*, 0^*]$

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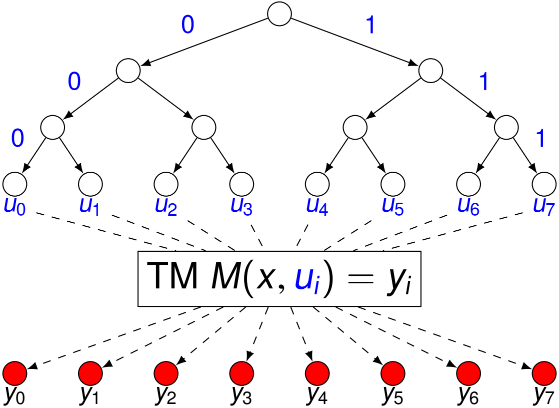

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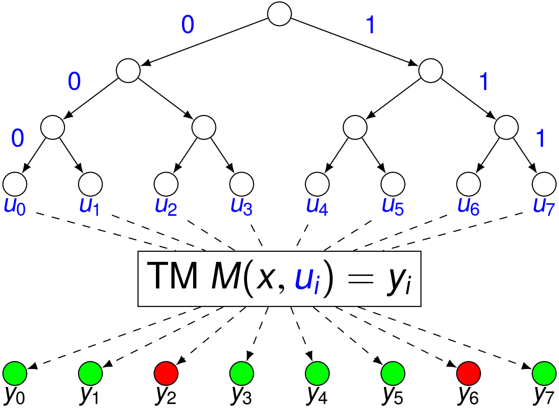

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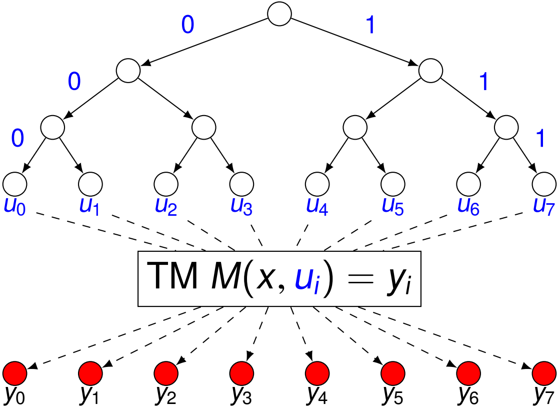

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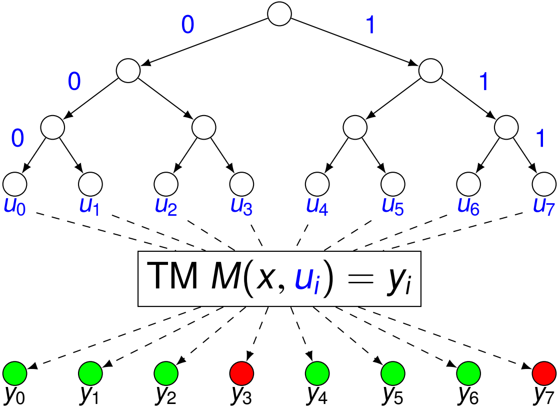

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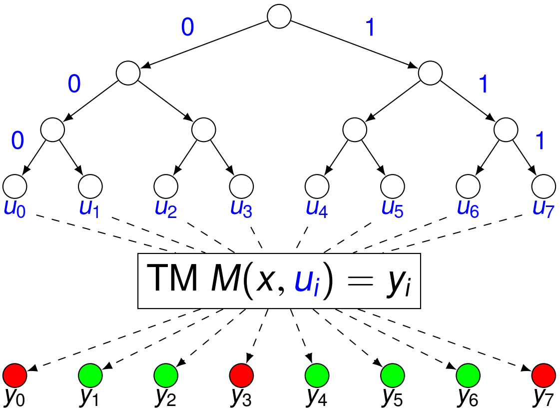

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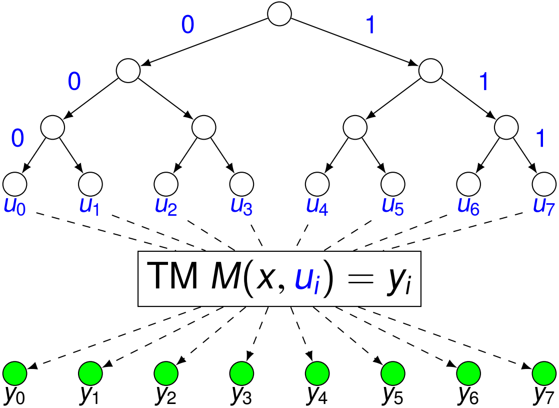

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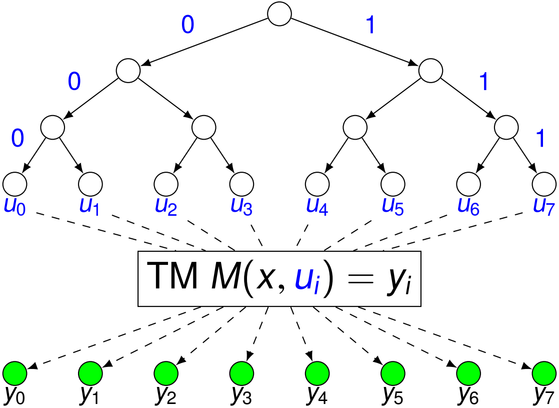

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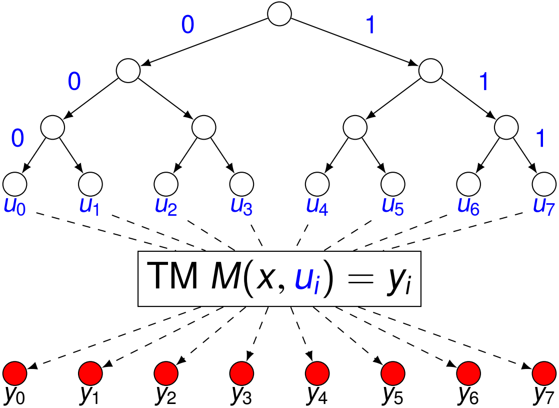

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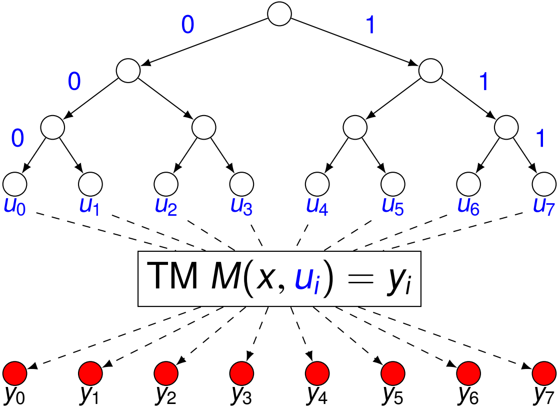

- What about **P**?
- $\mathbf{P} = \mathbf{C}[1(0+1)^*, 0(0+1)^*]$.
- Certificate $0 \ldots 0$ can always be used (compare this to **BPP**)