Complexity Theory

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Lecture 10

The polynomial hierarchy PH

Agenda

- ExactIndset, MinEqDNF, and bounded QBF
- Σ_i^p, Π_i^p, and **PH**
- properties of the polynomial hierarchy
- more examples

Exact independent set

Recall the independent set problem

$$\text{Indset} = \{\langle G, k\rangle \mid G \text{ has an independent set of size } k\}$$

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2. all other independent set have size at most k

(1) is a $\exists$ certificate (as in **NP**) while (2) is a $\forall$ certificate (as in **coNP**)!

Minimizing Boolean formulas

Let DNF be disjunctive normal form and $\equiv$ denote logic equivalence.

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What about $\overline{\text{MinEqDNF}}$?

Σ_2^p

Recall the certificate-based definitions of **NP** and **coNP**, where $q : \mathbb{N} \to \mathbb{N}$ is a polynomial, $x \in \{0,1\}^*$ and M is a polynomial-time, det. verifier.

NP $x \in L$ iff $\exists u \in \{0,1\}^{q(|x|)}.\ M(x,u) = 1$

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This class is called Σ_2^p.

Bounded QBF

Another natural problem within $\mathbf{\Sigma_2^p}$ is QBF with one alternation!

$$\Sigma_2\mathrm{SAT} = \{\exists \vec{u_1} \forall \vec{u_2}.\varphi(\vec{u_1}, \vec{u_2}) \mid \text{formula is true}\}$$

where $\vec{u_i}$ denotes a finite sequence of Boolean variables.

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Remarks

- in fact, Σ_2SAT is complete for $\mathbf{\Sigma_2^p}$
- more alternations lead to a whole hierarchy
- all of it is contained in **PSPACE**

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Definition

Definition (Polynomial Hierarchy)

For $i \geq 1$, a language $L \subseteq \{0,1\}^*$ is in $\mathbf{\Sigma_i^p}$ if there exists a polynomial-time TM M and a polynomial q such that

$$x \in L$$

if and only if

$$\exists u_1 \in \{0,1\}^{q(|x|)}.$$
$$\forall u_2 \in \{0,1\}^{q(|x|)}.$$
$$\ldots$$
$$Q_i u_i \in \{0,1\}^{q(|x|)}.$$
$$M(x, u_1, u_2, \ldots, u_i) = 1$$

where Q_i is $\exists$ if i is odd and $\forall$ otherwise.

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where Q_i is $\exists$ if i is odd and $\forall$ otherwise.

- the polynomial hierarchy is the set $\mathbf{PH} = \bigcup_{i \geq 1} \mathbf{\Sigma_i^p}$
- $\mathbf{\Pi_i^p} = \mathbf{co}\mathbf{\Sigma_i^p} = \{\overline{L} \mid L \in \mathbf{\Sigma_i^p}\}$

Generalization of NP and coNP

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- $\mathbf{NP} = \Sigma_1^p$ and $\mathbf{coNP} = \Pi_1^p$
- $\Sigma_i^p \subseteq \Pi_{i+1}^p \subseteq \Sigma_{i+2}^p$

Generalization of NP and coNP

- $\mathbf{NP} = \mathbf{\Sigma_1^p}$ and $\mathbf{coNP} = \mathbf{\Pi_1^p}$
- $\mathbf{\Sigma_i^p} \subseteq \mathbf{\Pi_{i+1}^p} \subseteq \mathbf{\Sigma_{i+2}^p}$
- hence $\mathbf{PH} = \bigcup_{i \geq 1} \mathbf{\Pi_i^p}$
- $\mathbf{PH} \subseteq \mathbf{PSPACE}$

Collapse

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Collapse

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This would imply that $\mathbf{\Sigma^p_i} = \mathbf{PH}$: the hierarchy collapses to the i-th level.

Most researchers believe that the hierarchy does not collapse.

Theorem (Collapse)

- *For every $i \geq 1$, if $\mathbf{\Sigma^p_i} = \mathbf{\Pi^p_i}$ then $\mathbf{PH} = \mathbf{\Sigma^p_i}$*
- *If $\mathbf{P} = \mathbf{NP}$ then $\mathbf{PH} = \mathbf{P}$, i.e. the hierarchy collapses to $\mathbf{P}$.*

Completeness

For each level of the hierarchy completeness is defined in terms of polynomial Karp reductions.

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Theorem (bounded QBF)

For each $i \geq 1$, Σ_iSAT is $\mathbf{\Sigma_i^p}$-complete, where Σ_iSAT is the language of true quantified Boolean formulas of the form

$$\exists \vec{u_1} \forall \vec{u_2} \ldots Q_i \vec{u_i}.\varphi(\vec{u_1}, \vec{u_1}, \ldots, \vec{u_i})$$

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Integer Expressions

An integer expression I is defined by the following BNF for binary numbers $\vec{b}$:

$$I ::= \vec{b} \mid I + I \mid I \cup I$$

The language $\mathcal{L}(I) \subseteq \mathbb{N}$ is defined by

- $\mathcal{L}(\vec{b}) = \{n\}$ where n is the natural number represented by $\vec{b}$
- $\mathcal{L}(I_1 + I_2) = \{n_1 + n_2 \mid n_i \in \mathcal{L}(I_i)\}$
- $\mathcal{L}(I_1 \cup I_2) = \mathcal{L}(I_1) \cup \mathcal{L}(I_2)$

Example: $\mathcal{L}(1 + (2 \cup (3 + 4))) = \{3, 8\}$

A set $M \subseteq \mathbb{N}$ is connected if for all $x, z \in M$ and every $x < y < z$ also $y \in M$.

A component of M is a maximal connected subset of M.

Integer Expressions

- membership of a number in the language of an integer expression: **NP**-complete
- integer expression inequivalence: $\mathbf{\Sigma_2^p}$-complete
- *Does* $\mathcal{L}(I)$ *have a component of size at least k*?: $\mathbf{\Sigma_3^p}$-complete

Regular Expressions

Consider regular expressions with union and concatentation only. In addition, we define an interleaving operator on words

$$x_1 x_2 \dots x_k \mid y_1 y_2 \dots y_k$$
$$=$$
$$x_1 y_1 x_2 y_2 \dots x_k y_k$$

where y_i can be strings of arbitrary length.

Regular expression equivalence for star-free expressions with interleaving is Π^p_2-complete.

Context-free languages

Consider context-free grammars defining unary languages.

- $\{\langle G_1, G_2\rangle \mid \mathcal{L}(G_1) \neq \mathcal{L}(G_2)\}$ is $\mathbf{\Sigma_2^p}$-complete
- note that for non-unary languages this problem is undecidable

Further Reading

Survey on complete problems for various levels of the hierarchy:

- *Schaefer and Umans* Completeness in the Polynomial-Time Hierarchy — A Compendium

What have we learnt?

- the polynomial hierarchy is a natural generalization of **NP** and **coNP**
- bounded alternation QBFs are complete problems for each level of the hierarchy
- in the limit – unbounded alternations – the hierarchy approaches **PSPACE**
- the hierarchy is widely believed not to collapse to any level