Complexity Theory

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Lecture 10–Part II

PH & co.

Agenda

- oracles
- oracles and **PH**
- relativization and **P** vs. **NP**
- alternation and **PH**

Minimizing Boolean formulas

Let DNF be disjunctive normal form and $\equiv$ denote logic equivalence.

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What if we can check equivalence of formulae for free?

Oracle

Definition

An oracle is a language A.
An oracle Turing machine M^A is a Turing machine that

1. has an extra *oracle* tape, and
2. can ask whther the string currently written on the oracle tape belongs to A and in a *single* computation step gets the answer.

P^A is a class of languages decidable by a polynomial-time oracle Turing machine with an oracle A; similarly **NP**^{A} etc.

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- We often write classes instead of the complete languages, e.g., $\mathbf{P}^{\mathbf{NP}} = \mathbf{P}^{\text{SAT}} = \mathbf{P}^{\mathbf{coNP}}$

Oracles and PH

Recall that

$$\Sigma_i\mathrm{SAT} = \{\exists \vec{u_1} \forall \vec{u_2} \cdots Q \vec{u_i}.\varphi(\vec{u_1}, \ldots, \vec{u_i}) \mid \text{formula is true } \}$$

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Theorem

For every i, $\Sigma_i^p = \mathbf{NP}^{\Sigma_{i-1}\mathrm{SAT}} = \mathbf{NP}^{\Sigma_{i-1}^p}$.

e.g. $\Sigma_3^p = \mathbf{NP}^{\mathbf{NP}^{\mathbf{NP}}}$

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Proof

$\subseteq$: easy

$\supseteq$ (here for i=2, i.e. $\mathbf{\Sigma_2^p} \supseteq \mathbf{NP}^{\mathrm{SAT}}$): Let φ_i denote the ith query

$x \in L \iff \exists c_1, \ldots, c_m, a_1, \ldots, a_k, u_1, \ldots, u_k \forall v_1, \ldots, v_k$ such that

TM accepts x using choices $c_1, \ldots, c_m$ and answers $a_1, \ldots, a_k$ AND

$\forall i \in [k]$ if $a_i = 1$ then $\varphi_i(u_i) = 1$ AND

$\forall i \in [k]$ if $a_i = 0$ then $\varphi_i(v_i) = 0$

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- But there exist oracles X and Y:
  - $\mathbf{P}^X \neq \mathbf{NP}^X$ (See Sipser p.378)
  - $\mathbf{P}^Y = \mathbf{NP}^Y$ (Proof: $\mathbf{NP}^{\mathrm{QBF}} \subseteq \mathbf{NPSPACE} \subseteq \mathbf{PSPACE} \subseteq \mathbf{P}^{\mathrm{QBF}}$)

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- Diagonalization has its limits!
  It is not sufficent to simulate computation,
  we must analyze them → e.g. cicuit complexity.

Agenda

- oracles ✓
- oracles and **PH** ✓
- relativization and **P** vs. **NP** ✓
- alternation and **PH**

Alternation

Recall that

- Σ_2SAT $= \{\exists \vec{u_1} \forall \vec{u_2}.\varphi(\vec{u_1}, \vec{u_2}) \mid$ formula is true $\}$ is $\mathbf{NP}^{\mathbf{coNP}}$-complete
- SAT $= \{\exists \vec{u_1}.\varphi(\vec{u_1}) \mid$ formula is true $\}$ is **NP**-complete
- VAL $= \{\forall \vec{u_1}.\varphi(\vec{u_1}) \mid$ formula is true $\}$ is **coNP**-complete

- $\exists \sim$ existential certificate $\sim$ there is an accepting computation
- $\forall \sim$ universal certificate $\sim$ all computations are accepting

Alternation

Definition

An alternating Turing machine is a Turing machine where

- states are partitioned into existential (denoted $\exists$ or $\vee$) and universal (denoted $\forall$ or $\wedge$),
- configurations are labelled by the type of the current state,
- a configuration in the computation tree is accepting iff
  - it is $\exists$ and some of its successors is accepting,
  - it is $\forall$ and all its successors are accepting.

We define **ATIME**, **ASPACE**, **AP**, **APSPACE** etc. accordingly.

Alternation and PH

Let $\Sigma_i P$ denote the set of languages decidable by ATM

- running in polynomial time,
- with initial state being existential, and
- such that on every run there are at most i maximal blocks of existential and of universal configurations.

Theorem

For all i, $\Sigma_i^p = \Sigma_i P$.

Power of alternation

Theorem

For $f(n) \geq n$, *we have*
$\mathbf{ATIME}(f(n)) \subseteq \mathbf{SPACE}(f(n)) \subseteq \mathbf{ATIME}(f^2(n))$.

For $f(n) \geq \log n$, *we have*
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Corollary:
$\mathbf{L} \subseteq \mathbf{AL} = \mathbf{P} \subseteq \mathbf{AP} = \mathbf{PSPACE} \subseteq \mathbf{APSPACE} = \mathbf{EXP} \subseteq \mathbf{AEXP} \cdots$

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- $\mathbf{ASPACE}(f(n)) \supseteq \mathbf{TIME}(2^{O(f(n))})$
  guess and check the tableaux of the computation
  (+ halting state on the left)

Further Reading

Alternation

- for a survey on alternation see *Chandra, Kozen, Stockmeyer* Alternation in Journal of the ACM 28(1), 1981.
- `http://portal.acm.org/citation.cfm?id=322243`

What have we learnt?

- the polynomial hierarchy can be defined in terms of certificates, recursively by oracles, or by bounded alternation
- diagonalization/simulation proof techniques have their limits
- alternation seems to add power:
  it moves us to the “next higher” class

Up next: time/space tradeoffs, **TISP**(f, g)