Complexity Theory

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Summer 2019

Partially based on slides by Jörg Kreiker

Lecture 1

Introduction

Agenda

- computational complexity and two problems
- your background and expectations
- organization
- basic concepts
- teaser
- summary

Computational Complexity

- quantifying the efficiency of computations
- not: computability, descriptive complexity, . . .
- computation: computing a function $f : \{0,1\}^* \to \{0,1\}^*$
  - everything else matter of encoding
  - model of computation?
- efficiency: how many resources used by computation
  - time: number of basic operations with respect to input size
  - space: memory usage

Dinner Party

Example (Dinner Party)

You want to throw a dinner party. You have a list of pairs of friends who do not get along. What is the largest party you can throw such that you do not invite any two who don't get along?

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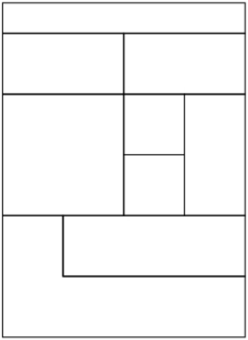
- largest party?
- naive computation
  - check all sets of people for compatibility
  - number of subsets of n element set is 2^n
  - intractable
- can we do better?
- observation: for a given set compatibility checking is easy

Map Coloring

Example (Map Coloring)

Can you color a map with three different colors, such that no pair of adjacent countries has the same color. Countries are adjacent if they have a non-zero length, shared border.



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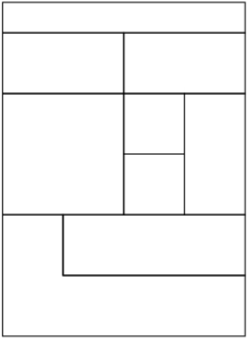

- naive algorithm: try all colorings and check
- number of 3-colorings for n countries: 3^n
- can we do better?
- observation: for a given coloring compatibilty checking is easy

What about you?

- What do you expect?
- What do you already know about complexity?
- Immediate feedback

Organization

- lecture in English
- course website:
  http://www7.in.tum.de/um/courses/complexity/SS19/
- concentrated into the first part of the semester, in 03.09.014
  - (reserved slot Monday 14-16)
  - Tuesday 10:05-11:35 and 12:25-13:55
  - Wednesday 8:25-9:55
  - Friday 12:05-13:35 and 14:00-15:30
- tutor: Mikhail Raskin
- weekly exercise sheets, not mandatory
- written or oral exam, depending on number of students
- bonus
  - several mini-tests during lectures (un-announced, cover 2-4 lectures)
  - self-assessment and feedback to us
  - if C is ratio of correct answers, exam bonus computed by

$$\frac{\lceil 5C - 1 \rceil}{2}$$

Literature

- lecture based on Computational Complexity: A Modern Approach by **Sanjeev Arora** and **Boaz Barak**
- book website:
  `http://www.cs.princeton.edu/theory/complexity/`
- useful links plus freely available draft
- lecture is self-contained
- more recommended reading on course website, e.g. Introduction to the Theory of Computation by **Michael Sipser**

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Prerequisites

- sets, relations, functions
- formal languages
- Turing machines
- graphs and algorithms on graphs
- little probability theory
- Landau symbols

Landau symbols

- characterize asymptotic behavior of functions (on integers, reals)
- ignore constant factors
- useful to talk about resource usage

Landau symbols

- characterize asymptotic behavior of functions (on integers, reals)
- ignore constant factors
- useful to talk about resource usage
- upper bound: $f \in \mathcal{O}(g)$ defined by
  $\exists c > 0.\ \exists n_0 > 0.\ \forall n > n_0.\ f(n) \leq c \cdot g(n)$
- dominated by: $f \in o(g)$ defined by $\forall \varepsilon > 0.\ \exists n_0 > 0.\ \forall n > n_0.\ \frac{f(n)}{g(n)} < \varepsilon$
- lower bound: $f \in \Omega(g)$ iff $g \in \mathcal{O}(f)$
- tight bound: $f \in \Theta(g)$ iff $f \in \mathcal{O}(g)$ and $f \in \Omega(g)$
- dominating: $f \in \omega(g)$ iff $g \in o(f)$

Intractability

Polynomial

versus

Exponential

- computations using exponential time or space intractable for all but the smallest inputs
- for a map with 200 countries: app. $2.66 \cdot 10^{95}$ 3-colorings
- atoms in the universe (wikipedia): $8 \cdot 10^{80}$
- computational complexity: tractable vs. intractable
- tractable: problems with runtimes $\bigcup_{p>0} \mathcal{O}(n^p)$
- intractable: problems with worse runtimes, e.g. $2^{\Omega(n)}$
- independent of hardware

What about our examples?

- dinner party problem tractable?
- map coloring problem tractable?
- lower bounds on time/space consumption
- upper bounds on time/space consumption
- which is harder?

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- each person a node, each relation an edge
- find a maximal set of nodes, such that no two nodes are adjacent

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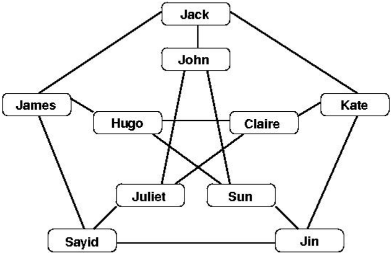

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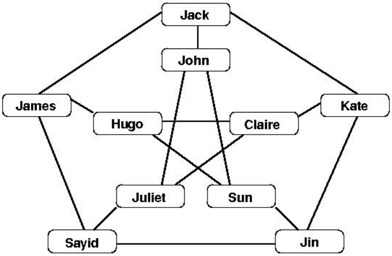

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- the independent set problem: Indset
- probably not tractable, no algorithm better than naive one known

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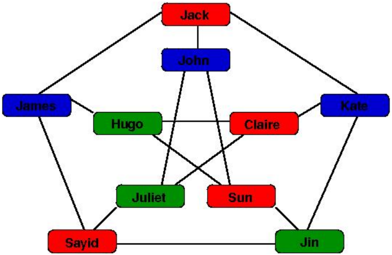

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- here: maximal independent set of size 4

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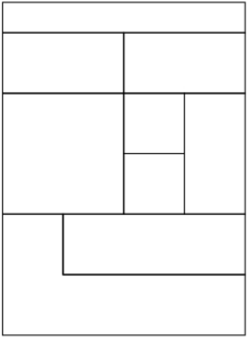

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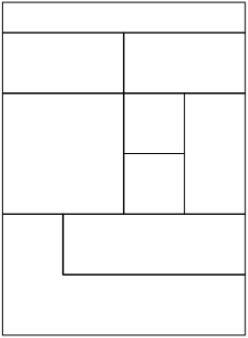

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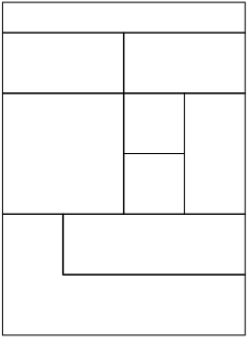

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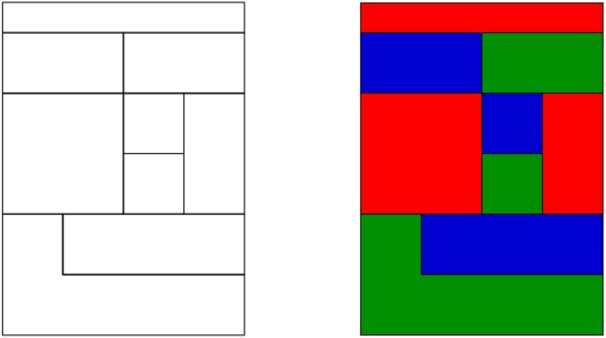

- really a graph problem
- each country a node, each border an edge
- color each node such that no two adjacent nodes have same color
- the three coloring problem: $3-$Coloring
- probably not tractable, no algorithm better than naive one known
- here: answer is yes

Bounds

- upper bounds
  - time (naive algorithm): $2^{\mathcal{O}(n)}$ for n persons/countries
  - space (naive algorith): $\mathcal{O}(n^p)$ for n persons/countries and a small p

Bounds

- upper bounds
  - time (naive algorithm): $2^{\mathcal{O}(n)}$ for n persons/countries
  - space (naive algorith): $\mathcal{O}(n^p)$ for n persons/countries and a small p
- lower bounds
  - very little known
  - difficult because of infinitely many algorithms
  - both problems could have a linear time and a logarithmic space algorithm
  - but not simultaneously

Which is harder?

- instead of tight bounds say which problem is harder
- $\Rightarrow$ reductions

Which is harder?

- instead of tight bounds say which problem is harder
- ⇒ reductions

IF there is an efficient procedure for *B* using a procedure for *A*

THEN *B* cannot be radically harder than *A*

notation: $B \leq A$

Which is harder?

- instead of tight bounds say which problem is harder
- $\Rightarrow$ reductions

IF there is an efficient procedure for B using a procedure for A

THEN B cannot be radically harder than A

notation: $B \leq A$

Applications:

- **IF**
  - there is an efficient procedure for problem A and
  - $B \leq A$

  THEN there is an efficient procedure for problem B
- **IF**
  - there is no efficient procedure for problem B and
  - $B \leq A$

  THEN there is no efficient procedure for problem A

3−Coloring $\leq$ Indset

How can we solve 3−Coloring using an algorithm to solve Indset?

$3-$Coloring $\leq$ Indset

How can we solve $3-$Coloring using an algorithm to solve Indset?

- triplicate the original graph (V, E) into $(V \times \{1, 2, 3\}, E')$ where

$$
\begin{aligned}
E' = & \{((v, i), (w, i)) \mid (v, w) \in E, i \in \{1, 2, 3\}\} \cup \\
& \{((v, i), (v, j)) \mid v \in V, i \neq j \in \{1, 2, 3\}\}
\end{aligned}
$$

efficient

$3-\text{Coloring} \leq \text{Indset}$

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efficient

- check whether there is an independent set of size $|V|$

assumed efficient

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How can we solve $3-$Coloring using an algorithm to solve Indset?

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efficient

- check whether there is an independent set of size $|V|$

assumed efficient

Need to ensure: procedure returns **yes** if and only if the graph is 3-colorable.

Polynomial certificates: NP

- whole class of problems can be reduced to Indset
- all of them have polynomially checkable certificates
- characterizes (in)famous class **NP**
- all problems in **NP** reducible to Indset makes Indset **NP**-hard.
- 3−Coloring also **NP**-hard
- no polynomial-time algorithms known for **NP**-hard problems
- not all problems have polynomial certificates, e.g. winning strategy in chess

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Lots of things to explore

- precise definition of computational model and resources
- problems with polynomial time checkable certificates
- space classes
- approximations
- hierarchies (polynomial, time/space tradeoffs)
- randomization
- parallelism
- average case complexities
- probabilistically checkable proofs
- (quantum computing)
- (logical characterizations of complexity classes)
- a bag of proof techniques

What have we learnt?

- polynomial $\sim$ tractable; exponential $\sim$ intractable
- lower bounds hard to come by
- reductions key to establish relations among (classes of problems)
- **NP**: polynomially checkable certificates
- Indset $\in$ **NP**, 3−Coloring $\in$ **NP**