

Computational Complexity – Homework 7

Discussed on 21.05.2019.

Exercise 7.1

Show that, if $\mathbf{NP} \subseteq \mathbf{BPP}$, then $\mathbf{RP} = \mathbf{NP}$.

Exercise 7.2

Show that

- (a) \mathbf{RP} , \mathbf{BPP} , and \mathbf{PP} are closed under \leq_p .

Remark: Recall that a class \mathbf{C} is closed under \leq_p if $A \leq_p B \wedge B \in \mathbf{C} \Rightarrow A \in \mathbf{C}$.

- (b) \mathbf{RP} and \mathbf{BPP} are closed under intersection and union.

Exercise 7.3

A *probabilistic alternating* Turing machine (short: PATM) is a tuple $(Q_{\frac{1}{2}}, Q_{\exists}, \Gamma, \delta_0, \delta_1)$ where

- $Q := Q_{\frac{1}{2}} \cup Q_{\exists}$ is the set of control states. ($Q_{\frac{1}{2}}$ and Q_{\exists} are required to be disjoint.)
- Γ is the alphabet.
- δ_0, δ_1 are two transition functions.

A run of a PATM $M = (Q_{\frac{1}{2}}, Q_{\exists}, \Gamma, \delta_0, \delta_1)$ on a given input x is simply a run by the underlying NDTM defined by $(Q_{\frac{1}{2}} \cup Q_{\exists}, \Gamma, \delta_0, \delta_1)$. In particular, M runs in time $T(n)$ if every run on input x takes time at most $T(|x|)$, i.e., the computation tree of M on input x has height at most $T(|x|)$. (Recall the inductive definition of configuration tree: starting from the initial configuration on input x (the root), every inner node of the tree is a non-halting configuration c of M which has exactly two children $\delta_0(c)$ and $\delta_1(c)$, even if $\delta_0(c) = \delta_1(c)$.)

The intuition of a PATM is that it combines randomization with nondeterminism: in a configuration with a control state contained in Q_{\exists} a PATM basically explores both possible successors in parallel, while in a configuration with control state in $Q_{\frac{1}{2}}$ it chooses one of the two possible successors uniformly at random. More formally, the probability that M accepts x ($\Pr[M(x) = 1]$) is then defined by labeling the computation tree bottom-up as follows:

- A leaf is labeled by 1 if it corresponds to an accepting configuration, otherwise it is labeled by 0.
- An inner node which corresponds to a control state from $Q_{\frac{1}{2}}$ is labeled by the average of the labels of its two children;
- while an inner node corresponding to a control state from Q_{\exists} is labeled by the maximum of its two children.

The label of the root of the computation tree of M on input x is then the probability that M accepts x , short $\Pr[M(x) = 1]$. Similarly, $\Pr[M(x) = 0] := 1 - \Pr[M(x) = 1]$.

- (a) Show that for every poly-time PATM M there is a poly-time PATM N s.t.:

- $\Pr[M(x) = 1] = \Pr[N(x) = 1]$ for all $x \in \{0, 1\}^*$.
- Every run of N on a given input x takes time exactly $2|x|^k$ for some $k > 0$.
- Every inner node with control state in $Q_{\frac{1}{2}}$ (Q_{\exists}) has only children with control state in Q_{\exists} ($Q_{\frac{1}{2}}$).

- (b) Let $M = (Q_{\exists}, Q_{\forall}, \Gamma, \delta_0, \delta_1)$ be a poly-time ATM deciding the language L . We can reinterpret M also as a PATM by setting $Q_{\frac{1}{2}} := Q_{\forall}$. Show that

$$x \in L \Leftrightarrow \Pr[M(x) = 1] = 1.$$

- (c) The class \mathbf{APP} is defined as follows:

A language L is contained in \mathbf{APP} if there is a poly-time PATM M s.t.

$$x \in L \Leftrightarrow \Pr[M(x) = 1] \geq 3/4.$$

- Show that $\mathbf{APP} \subseteq \mathbf{PSPACE}$ by adapting the \mathbf{PSPACE} -algorithm for deciding QSAT.

- Show that $\mathbf{PSPACE} \subseteq \mathbf{APP}$ by adapting the proof of $\mathbf{NP} \subseteq \mathbf{PP}$ given in the lecture.

Hint: Recall that $\mathbf{AP} = \mathbf{PSPACE}$, i.e., for every $L \in \mathbf{PSPACE}$ there is a poly-time alternating Turing machine deciding L . Now copy the construction from the proof of $\mathbf{NP} \subseteq \mathbf{PP}$ in order to obtain from a poly-time ATM a poly-time PATM M with $x \in L \Leftrightarrow \Pr[M(x) = 1] \geq 3/4$.

- (d) The class \mathbf{ABPP} is defined as follows:

A language L is contained in \mathbf{ABPP} if there is a poly-time PATM M s.t.

$$x \in L \Rightarrow \Pr[M(x) = 1] \geq 3/4 \text{ and } x \notin L \Rightarrow \Pr[M(x) = 1] \leq 1/4.$$

Obviously, we have $\mathbf{ABPP} \subseteq \mathbf{APP}$.

- Show that $\mathbf{ABPP} = \mathbf{IP} = \mathbf{APP} = \mathbf{PSPACE}$.

Hint: You already know \mathbf{ABPP} from the lecture by some other name.

- (e) Assume we extend the definition of PATMs by partitioning the control states into three classes $Q_{\frac{1}{2}}, Q_{\exists}, Q_{\forall}$; the acceptance probability $\Pr[M(x) = 1]$ is defined as above where the value of a node corresponding to a control state of Q_{\forall} is defined to be the minimum of the values of its two children. Call such a Turing machine a PAATM.

- Using PAATMs define the complexity classes \mathbf{AAPP} and \mathbf{AABPP} analogously to \mathbf{APP} and \mathbf{ABPP} .

Discuss how these relate to \mathbf{APP} , \mathbf{PP} , \mathbf{ABPP} , \mathbf{BPP} , \mathbf{AP} , \mathbf{PSPACE} , \mathbf{IP} , \mathbf{AM} .