Technische Universität München (I7)
Prof. J. Křetínský / Dr. M.Raskin

Summer term 2019
6 May 2019

Solution

Computational Complexity – Homework 4

Discussed on 8 May 2016.

Exercise 4.1

Let us denote $\mathbf{DP} = \{L \mid \exists M, N \in \mathbf{NP} : L = M \setminus N\}$ the class of languages that are diferences of two NP languages.

(a) Show that $C = \{\langle G_1, k_1, G_2, k_2\rangle \mid G_1$ has a k_1-clique and G_2 does not have any k_2-clique$\}$ is DP-complete.

(b) Show that $MAX - CLIQUE = \{\langle G, k\rangle \mid$ the largest clique of G is of size exactly $k\}$ is DP-complete.

(c) It is unknown whether $MAX - CLIQUE$ is in $\mathbf{NP}$. Show that if $\mathbf{P} = \mathbf{NP}$ then $MAX - CLIQUE$ is in $\mathbf{NP}$ and a largest clique can be found in polynomial time.

Solution:

(a) Let us define

$$C_1 := \{\langle G_1, k_1, G_2, k_2\rangle \mid G_1 \text{ has a } k_1\text{-clique and } G_2 \text{ is any graph and } k_2 \geq 0\}$$

and

$$C_2 := \{\langle G_1, k_1, G_2, k_2\rangle \mid G_2 \text{ has a } k_2\text{-clique and } G_1 \text{ is any graph and } k_1 \geq 0\}$$

Both these languages are NP-complete as we have already seen (they are both just the clique problem with unconstrained extra data attached to each)

Then $C = C_1\ C_2$ and so $C \in DP$.

Now we show that C is DP-hard. Suppose that $L \in DP$. It must be the case that $L = L_1\ L_2$ where $L_1, L_2 \in NP$ (by definition). By the NP-completeness of the clique problem, we must thus have polynomial-time reductions f_1 and f_2 from L_1 and L_2 to respectively to the clique problem.

Thus we can define a polynomial-time reduction f from L to C by:

$$f(w) := \langle f_1(w), f_2(w)\rangle$$

This is clearly computable in polynomial time (since both $f_1(w)$ and $f_2(w)$ are), and moreover we have $w \in L$ iff $f(w) \in C$ since $w \in L$, iff $w \in L_1$ and $w \notin L_2$, iff $f_1(w) \in CLIQUE$ and $f_2(w) \notin CLIQUE$ (since f_1 and f_2 are reductions to $CLIQUE$), iff $f(w) \in C$.

(b) To see that MAX-CLIQUE is in DP, observe that the problem $CLIQUE{+} := \{\langle G, k\rangle \mid G$ has a $k+$ 1clique$\}$ is in NP. It is then the case that MAX-CLIQUE is equal to $CLIQUE\ CLIQUE{+}$.

To show that MAX-CLIQUE is DP-hard, we show that there is a polynomial time reduction from C (in the previous part) to MAX-CLIQUE.

Consider a tuple $\langle G_1, k_1, G_2, k_2\rangle$. We define a pair $\langle G, k\rangle$ that can be computed from the tuple in polynomial time such that $\langle G_1, k_1, G_2, k_2\rangle \in C$ iff $\langle G, k\rangle \in MAX - CLIQUE$.

Let N_1, N_2 be respectively the node sets of G_1 and G_2 and E_1, E_2 their respective edge relations. Let $K_{k'}$ be the clique of size k'.

Consider first the graph $G_1' := (N_1', E_1')$, where $N_1' := [1, k_1] \times N_1$ and $E_1' := \{((i, u), (i + 1, v) \mid i \in [1, k_1)$ and $(u, v) \in E_1\}$. By construction, no clique of G_1' can be bigger than k_1, and it will have a clique of size k_1 iff G_1 also has a clique of size k_1.

For $r \in \mathbb{N}$, we extend G_1' to a graph G_1^r by adding an instance of K_r and then an edge from each node in this instance of K_r to each node in the original G_1'. The graph G_1^r will now have a clique of size $k_1 + r$ iff G_1 has a clique of size k_1, and moreover no clique of G_1^r can be bigger than $k_1 + r$. That is, G_1 has a clique of size k_1, iff the maximum-sized clique in G_1^r has size $k_1 + r$.

We now define the graph G_2^r (i) first in a similar way to G_1^r, replacing k_1 with k_2, N_1 with N_2 and E_1 with E_2, and then in addition (ii) adding a fresh disjoint instance K_{k_2+r-1}.

It will then be the case that G_r^2 has a $(k_2 + r)$-clique iff G_2 has a k_2-clique. Moreover, it is certain that g_2^r has a $k_2 + r - 1$ sized clique and that it has no clique larger than $k_2 + r$. Thus the maximal clique of G_r^2 has size $k_2 + r$ iff G_2 has a k_2-clique.

If $k_1 > k_2$ we can thus take $G' := G_1^0 \times G_2^{k_1-k_2}$, and otherwise $G' := G_1^{k_2-k_1} \times G_2^0$.

Exercise 4.2

(a) Assume that **P**=**NP**. Show that then **EXP**=**NEXP**.

Remark: Assume that L is decided by some TM running in time $T(n)$ with $T(n)$ time-constructible and $T(n) \in \mathcal{O}(2^{n^c})$ for some $c \geq 1$. Show that then

$$L_{\text{pad}} := \{x10^{T(|x|)}1 \mid x \in L\} \in \mathbf{NP}.$$

*(b) Show that also **EXP**=**NEXP** if only *every unary* **NP**-language is also in **P**.

Remark: For $x \in \{0,1\}^*$ let $\langle x\rangle$ be the natural number represented by x assuming lsbf. Given a language L which is decided in time $T(n)$ (with $T(n)$ time-constructable) show that

$$L_{\text{upad}} = \{1^{\langle x10^{|T(n)|}1\rangle} \mid x \in L\} \in \mathbf{NP}$$

with $|T(n)|$ ($\approx \lceil \log T(n)\rceil$) the length of the lsbf representation of $T(n)$.

Solution: Let $L \in \mathbf{NEXP}$ be decided the NTM by N in time $T(n) \in \mathcal{O}(2^{n^c})$ for some $c \geq 1$. Further, let M_T be the TM that computes $x \mapsto \text{lsbf}(T(|x|))$ in time $T(|x|)$.

We claim $L_{\text{pad}} \in$ NP: (If a "check" fails, we reject the input.)

- On input $y = 1^m$ first compute $w = \text{lsbf}(m)$.
- Then check that $w = z10^k1$ for some $z \in \{0,1\}^*$ and $k \in \mathbb{N}$.
- Next, simulate M_T on input z for exactly 2^{k+1} steps and check that the halting configuration is reached.

  Note that
  $$m = \langle w\rangle = \langle z10^k1\rangle \geq \langle 0^k1\rangle = 2^{k+1}.$$
- As M_T terminates, its output is $\text{lsbf}(T(|z|))$. Check that $k = |T(|z|)|$.
- Now simulate N on z for exactly 2^{k+1} steps. As
  $$2^{k+1} = 2^{|T(n)|+1} \geq T(n)$$
  the simulation reaches the halting configuration and therefore decides whether $z \in L$ or not.

As we assume that every unary language in **NP** is also in **P**, we also find a TM M which decides L_{pad} in polynomial time. From M we obtain a TM M' which decides L in **EXP**:

- For input x ($n = |x|$) first compute $\text{lsbf}(T(n))$ in time $T(n)$.
- Then generate $w = x10^{|T(n)|}1$ in time $n + 2 + |T(n)| = n + 2 + \lceil \log T(n) \rceil$.
- Finally, generate $y = 1^{\langle w \rangle}$. Note that

$$|y| = \langle w \rangle = \langle x10^{|T(n)|}1 \rangle \leq \langle 0^{|w|}1 \rangle = 2^{|w|+1} = 2^{2+n+|T(n)|+1} \leq 2^{n+4} \cdot T(n).$$

- Now use M' to decide whether $y \in L_{\text{pad}}$ or not.

Exercise 4.3

Is there a language in $\mathbf{DSPACE}(2^{2^{2^{\mathcal{O}(n)}}})$ that is not in **EXPSPACE** and not **NP**-hard (assuming $\mathbf{P} \neq \mathbf{NP}$)?

Solution: The idea of the solution is to note that the hardest language in $\mathbf{DSPACE}(2^{2^{2^{\mathcal{O}(n)}}})$ stays hard even after unary encoding.

Then language of unary encodings of such numbers n that the n-th Turing machine stops on empty input using no more than $2^{2^{2^n}}$ cells cannot be in **EXPSPACE** because the diagonal construction applies here.

But if a unary language is **NP**-hard, then $\mathbf{P} = \mathbf{NP}$.

Exercise 4.4

- Is the following problem in $\mathbf{DTIME}(2^{\mathcal{O}(n)})$?

  A function $f : \{1, \ldots, n\} \times \{1, \ldots, n\} \to \{1, \ldots, n\}$ is given as a table of values. Is there a sequence of n values $x_1, \ldots, x_n \in \{1, \ldots, n\}$ such that $f(x_1, f(x_2, \ldots f(x_{n-1}, x_n) \ldots)) = n$?

- Is the following problem in $\mathbf{DTIME}(2^{\mathcal{O}(n)})$?

  Multiplying an $n \times m$ matrix by an $m \times l$ matrix yields an $n \times l$ matrix and is implemented with time complexity $C \times m \times n \times l$ (the constant C is known).

  Given the number of steps T and the number of matrices k (both written in unary), determine whether there are k matrices that can be multiplied in T steps but not $0.9T$ steps.

Solution:

Exercise 4.5

Say that A is *linear-time reducible* to B if there is function f computable in time $\mathcal{O}(n)$ such that $x \in A \Leftrightarrow f(x) \in B$.

- Show that there is no P-complete problem w.r.t. linear-time reductions.

Hint: Use the time hierarchy theorem for **DTIME**.

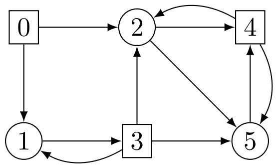
Exercise 4.6

A *two-person game* consists of a directed graph $G = (V_0, V_1, E)$ (called the *game graph*) whose nodes $V := V_0 \cup V_1$ are partitioned into two sets and a *winning condition*. We assume that every node $v \in V$ has a successor. The two players are called for simplicity player 0 and player 1. A *play* of the two is any finite or infinite path $v_1 v_2 \ldots$ in G where v_1 is the starting node. If the play is currently in node v_i and $v_i \in V_0$, then we assume that it is the turn of player 0 to choose v_{i+1} from the successors of v_i; if $v_i \in V_1$, player 1 determines the next move. The winning condition defines when a play is won by player 0. E.g.:

- In a *reachability game* the winning condition is simply defined by a subset $T \subseteq V_0 \cup V_1$ (*targets*) of the nodes of G, and a play is won by player 0 if it visits T within $n-1$ moves (where n is the total number of nodes of G). Hence, player 1 wins a play if he can avoid visiting T for at least $n-1$ moves.
- In a *revisiting game* player 0 wins a play $v_1 v_2 \ldots$ if the first node v_i which is visited a second time belongs to player 0, i.e., $v_i \in V_0$; otherwise player 1 wins the play.

We say that *player i wins node s* if he can choose his moves in such a way that he wins any play starting in s.

Example: Consider the following game graph where nodes of V_0 (V_1) are of circular (rectangular) shape:



In the reachability game with $T = \{5\}$ player 0 can win node 4: if player 1 moves from 4 to 5, player 0 immediately wins; if player 1 moves from 4 to 2, then player 0 can win again by moving from 2 to 5. On the other hand, player 1 can win node 0 by choosing to always play from 0 to 1 and from 3 to 1.

In the revisiting game played on the same game graph, player 0 can win node 2: he moves from 2 to 5 and then on to 4; no matter how player 1 then chooses to move, the play will end in an already visited node which belongs to player 0. Player 1 can e.g. win node 3 by simply moving to node 1.

(a) Consider a reachability game:

Show that one can decide in time polynomial in $\langle G, s, T\rangle$ if player 0 can win node s.

Hint: Starting in T compute the set of nodes from which player 0 can always reach T no matter how player 1 chooses his moves.

(b) Consider a revisiting game and the decision problem: for a given game graph G and node s determine whether player 0 can win s.

Show that this decision problem is in **PSPACE**.

(c) Show that this decision problem is **PSPACE**-complete.

Remarks:

- A game is called *determined* if every node if won by one of the two players.

  Are reachability, resp. revisiting games determined?

- Assume that we change the definition of reachability game by dropping the restriction on the number of moves, i.e., player 0 wins a play if the play eventually reaches a state in T.

  Does this change the nodes player 0 can win for a given game graph?

Solution:

(a) Let

$$A_0(X) := \{v \in V_0 \mid vE \cap X \neq \emptyset\} \cup \{v \in V_1 \mid vE \subseteq X\}.$$

and

$$W_0 := \bigcup_{k \geq 0} A_0^k(T).$$

Note that $A_0(X)$ can be computed in time $|V||E|$ and W_0 in time $|V|^2|E|$ as we can include at most $|V|$ many nodes.

Induction on k shows that player 0 can win any node in $A_0^k(T)$ by simply playing to some node in $A_0^{k-1}(T)$. Any such play has trivially length at most $n-1$ (assuming T is not empty).

Consider any node $v \notin W_0$ and consider any play from v which reaches T. There is some smallest i such that $v_i \notin W_0$ and $v_{i+1} \in W_0$. As player 0 can win anny node in W_0, we can assume that the remaining play stays in W_0. If v_i was in V_0, then by definition of $A_0(W)$ we also would have $v_i \in A_0(W_0) = W_0$. So, $v_i \in V_1 \cap W_0$. Hence, player 1 can find a successor of v_i which is not contained in W_0, i.e., player 1 can always evade entering $W_0 \supset T$.

W_0 is therefore the set of nodes which player 0 can win and $V \setminus W_0$ is the set of nodes which player 1 can win. In particular, reachability games are determined. Note that we didn't really use the restriction

(b) v is won by player 0 iff we do not find a play which is won by player 1. Any play has length at most n. So, for a given node v, we can enumerate all possible plays in polynomial space and, hence, decided whether v is won by player 0.

One can also show that the revisiting game is determined: Consider the enlarged game graph, where nodes correspond to plays of length at most n. We have an edge from $v_1 v_2 \dots v_k$ to $v_1 v_2 \dots v_k v_{k+1}$ iff $(v_k, v_{k+1}) \in E$. Set now as target set the sequences which revisit a node of V_0 for the first time. Then player 0 wins v in the revisiting game on the original game graph iff he wins v in the reachability game on the enlarged game graph with target set T. As the reachability game is determined, the revisiting game is determined too.