**Technische Universität München (I7)**
**Prof. J. Křetínský / Dr. C.H. Broadbent**

**Summer term 2016**
**11. Mai 2016**

# Computational Complexity – Homework 5

Discussed on 16.5.2016.

## Exercise 5.1

(a) Show that for any $L \in \mathbf{PSPACE}$ there is single-tape TM $M$ (which may also write on its input tape) which decides $L$ also in polynomial space.

(b) Show that it is **PSPACE**-complete to decide if a given word $w$ can be derived by a given context-sensitive grammar $G$, i.e.,

$$\textsc{ConSens} := \{\langle G, w\rangle \mid \text{if } G \text{ is a context-sensitive grammar and } w \in L(G)\}.$$

## Exercise 5.2

Prove that $\mathbf{EXPTIME} = \mathbf{APSPACE}$.

[*Hint:* For the $\subseteq$ direction consider breaking the work tape(s) into exponentially many segments which are then independently simulated in polynomial space. Use alternatation to coordinate these simulations.]

*Remark:* We can also show that $\mathbf{P} = \mathbf{AL}$ (alternating logarithmic space).

## Exercise 5.3

We will revisit two-player graph games, but this time we will not bound the number of moves in a play, and even allow the number of moves to be infinite.

A *game graph* is a structure $\langle V, E, V_0, V_1, v\rangle$ where $\langle V, E\rangle$ is a finite directed graph, and $V_0, V_1$ is a partition of the vertices $V$. Moreover $v \in V$ is the *initial node*.

Consider a sequence of nodes $(u)_{u\in I}$ where $I \subseteq \mathbb{N}$ is a downward closed index set (which may or may not be infinite) for the sequence. Such a sequence is called a *partial play* if (i) $u_0 = v$, and (ii) $(u_i, u_{i+1}) \in E$ for all $i + 1 \in I$. A partial play is called a *play* if either $I = \mathbb{N}$, or it is a finitely long play $u_0, \dots, u_k$ such that there is no edge $(u_k, u) \in E$ for any $u \in V$.

Two players (player 0 and player 1) between them construct a partial play. The partial play begins with $v$. If a partial play $v_0, \dots, v_i$ has been constructed, and $v_i \in V_j$, and there exists $u \in V$ such that $(v_i, v) \in E$, then player $j$ *must* choose the next node $v_{i+1}$ in the partial play such that $(v_i, v_{i+1}) \in E$. The partial play is extended no further if no such move exists.

Thus after either finitely many or infinitely many moves the two players will have constructed a partial play that is a play.

We consider three different types of game that are distinguished by their *winning conditions* $W$. Given a play $\sigma$, we write $Occ(\sigma)$ for the set of nodes occurring at least once in $\sigma$, and $Inf(\sigma)$ for the set of nodes occurring infinitely often in $\sigma$ (which will in particular be empty if $\sigma$ is only finitely long).

- In a *reachability game* $W \subseteq V$ and player 0 wins a play $\sigma$ if $W \cap Occ(\sigma) \neq \emptyset$.
- In a *Rabin game*, $W$ is a set of pairs of the form $(F, I)$ where $F, I \subseteq V$. Player 0 wins the play $\sigma$ if there exists $(F, I) \in W$ such that $F \cap Inf(\sigma) = \emptyset$ and $I \cap Inf(\sigma) \neq \emptyset$.
- In a *Müller game*, $W = \langle C, \mathcal{C}, \chi \rangle$ where $C$ is a finite set of colours, $\mathcal{C} \subseteq 2^C$, and $\chi : V \to C$. Player 0 wins a play $\sigma$ if $\chi(Inf(\sigma)) \in \mathcal{C}$.

The decision problem associated with a particular type of game is the set containing elements $\langle \mathcal{G}, W \rangle$ where $\mathcal{G}$ is a game graph, $W$ is an appropriate winning condition, and Player 0 can play in such a way that a play winning for Player 0 always results regardless of how Player 1 moves.

(a) Prove that the decision problem for reachability games is **P**-hard. (Remember that logarithmic space reductions must be used for this). For this take it as given that $\mathbf{AL} = \mathbf{P}$.

[*Remark:* It is possible to see that the version of reachability games defined in the previous problem sheet are equivalent to those defined above. Thus in fact reachability games are **P**-complete.]

(b) Prove that the decision problem for Rabin games is **NP**-complete.

[*Hint:* For hardness reduce from 3-SAT. Make Player 0 'prove' that they know some satisfying assignment. Allow Player 1 to 'interrogate' player 0's knowledge of such an assignment. Using the winning condition to ensure that for *some* literal player 0 is *eventually* consistent should suffice to allow Player 1 to successfully catch out Player 0 if no satisfying assignment exists. ]

(c) Prove that the decision problem for Müller games is **PSPACE**-complete.

[*Hint:* For hardness reduce from QBF. Observe that Rabin conditions can be (in polynomial time) translated into Müller conditions. Note further that the *complement* of a Rabin condition can also be so translated. You might also find it helpful to work with a slight generalisation of Müller games allowing one to have a Müller game equivalent of adding quantifiers to the front of a propositional formula. ]