Complexity Theory

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Based on slides by Jörg Kreiker

Lecture 9

NL

Agenda

- about logarithmic space
- paths . . .
- . . . and the absence thereof
- Immerman-Szelepcsényi and others

What can one do with logarithmic space?

In essence an algorithm can maintain a constant number of

- pointers into the input
  - for instance node identities (graph problems)
  - head positions
- counters up to input length

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Examples:

- **L**: basic arithmetic
- **NL**: paths in graphs

Technical issues

- space usage refers to work tapes only
- read-only input and write-once output is allowed to use more than $\log n$ cells
- write-once: output head must not move to the left
- logspace reductions (because polynomial time-reductions too powerful)

Logspace reductions

Recall Exercise 2.3!

Definition (logspace reduction)

Let $L, L' \subseteq \{0,1\}^*$ be languages. We say that L is logspace-reducible to L', written $L \leq_{log} L'$ if there is a function $f : \{0,1\}^* \rightarrow \{0,1\}^*$ computed by a deterministic TM using logarithmic space such that $x \in L \Leftrightarrow f(x) \in L'$ for every $x \in \{0,1\}^*$.

- $\leq_{log}$ is transitive
- $C \in \mathbf{L}$ and $B \leq_{log} C$ implies $B \in \mathbf{L}$
- **NL**-hardness and **NL**-completeness defined in terms of logspace reductions

Read-once Certificates

Similar to **NP**, also **NL** has a characterization using certificates

Theorem (read-once certificates)

*$L \subseteq \{0,1\}^*$ is in **NL** iff there exists a det. logspace TM M (verifier) and a polynomial $p : \mathbb{N} \to \mathbb{N}$ such that for every $x \in \{0,1\}^*$*

$$x \in L \text{ iff } \exists u \in \{0,1\}^{p(|x|)}.M(x,u) = 1$$

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- example: path in a graph is a read-once certificate

$\Rightarrow$ certificate is sequence of choices

$\Leftarrow$ certificate is guessed bit-wise (it cannot be stored)

- exercise: if read-once is relaxed, one arrives at **NP**

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Recall the language Path in directed graphs defined as

$$\{\langle G, s, t\rangle \mid \exists \text{a path from } s \text{ to } t \text{ in directed graph } G\}$$

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We have seen in Lecture 3 that Path $\in$ **NL** by guessing a path:

- non-deterministic walks on graphs of n nodes
- if there is a path, it has length $\leq n$
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In fact we even have:

Theorem (Path)

Path *is* **NL***-complete.*

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Proof

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- on input $x \in \{0,1\}^n$ reduction f outputs configuration graph $G(M,x)$ of size $2^{O(\log n)}$ by counting to n
- there exists a path from C_{start} to C_{accept} in $G(M,x)$ iff M accepts x
- path itself can be used as read-once certificate

More path problems

- many natural problems correspond to path (reachability) problems
- the word problem for NFAs: $\{\langle A, w\rangle \mid w \text{ is accepted by NFA } A\}$
- cycle detection/connected components in directed graphs
- $\overline{\text{2SAT}} \in$ **NL**

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- $\overline{\text{2SAT}} \in$ **NL**
  - $x \vee y$ equivalent to $\neg x \implies y$ equivalent to $\neg y \implies x$
  - yields an implication graph (computable in logspace)
  - unsatisfiable iff there exists a path $x \to \overline{x} \to x$ in implication graph for variable x

Certificates for absence of paths?

- recall the open problem **NP** $=$ **coNP**?
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What about absence of paths from s to t in graph G with n nodes named 1,. . . ,n?

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Certificate is certificate for non-membership in C_n!

Its size is polynomial in number of nodes and read-once!

NL = coNL

We have just argued the existence of polynomial read-once certificates for absence of paths.

Theorem (Immerman-Szelepcsényi)

NL = coNL.

What have we learnt?

- space classes closed under complement
  - so are context-sensitive language (see exercises)
- analogous results for time complexity unlikely
- space classes beyond logarithmic closed under non-determinism
- **NL** is all about reachability
- $\overline{\text{2SAT}}$ is in **NL** and thus 2SAT (in fact, hard for **NL**)
- **NL** has polynomial read-once certificates
- logarithmic space $\sim$ constant number of pointers and counters

Up next: the polynomial hierarchy **PH**

Further Reading

- paths in undirected graphs is in **L**
  - *Omer Reingold* Undirected ST-Connectivity in Log-Space, STOC 2005
  - available from `http://www.wisdom.weizmann.ac.il/~reingold/publications/`
- an alternative characterization of **NL** by reachability is at the heart of descriptive complexity (later this course)
  - **NL** is first-order logic plus transitive closure
  - *Neil Immerman*, Descriptive Complexity, Springer 1999.