Complexity Theory – Homework 9

Discussed on 30.06.2010.

Exercise 9.1

Given an undirected graph $G = (V, E)$ we call $C \subseteq V$ a *(vertex) cover* of G if

$$\forall (u, v) \in E \,: u \in C \vee v \in C.$$

In the following, we want to study the relation between several decision and function problems related to vertex cover. These are:

- $\text{VC}_D := \{\langle G, k\rangle \mid G$ has a vertex cover of size at most $k\}$.
- $\text{MinVC}_D := \{\langle G, k\rangle \mid G$ has a minimal vertex cover of size exactly $k\}$.
- $\text{MinSolVC}_D := \{\langle G, C\rangle \mid C$ is a minimal vertex cover of $G\}$.
- Calculate the minimal size $\text{minVC}(G)$ of a vertex cover of G.
- Calculate a minimal vertex cover $\text{MinVC}(G)$ of G.

Show:

(a) $\text{MinSolVC}_D \leq_p \text{MinVC}_D$.

(b) $\text{MinSolVC}_D \leq_p \overline{\text{VC}}_D$ (see Ex. 4.1).

(c) MinVC_D is **DP**-complete (see Ex. 6.3).

(d) $\text{VC}_D \leq_p \text{MinVC}_D$ and $\overline{\text{VC}}_D \leq_p \text{MinVC}_D$.

Remark: You only have to show that such reductions exist.

(e) Assume that $\text{minVC}(G)$ can be calculated in time $T(|G|)$.

Give bounds on the time needed to decide MinVC_D and MinSolVC_D, resp. calculate $\text{MinVC}(G)$.

In particular, show that, if $T(n)$ is polynomial, then so are the other bounds.

(f) Analogously to (e), give time bounds on the considered problems assuming that $\langle G, k\rangle \in \text{MinVC}_D$ (resp. $\langle G, k\rangle \in \text{VC}_D$) can be decided in time $T(|G|)$.

(g) If $\text{MinVC}_D \leq_p \text{MinSolVC}_D$, then $\mathbf{PH} \subseteq \mathbf{\Sigma}_1^p$.

Exercise 9.2

Let $\Phi = \{\phi_1, \ldots, \phi_m\}$ be a set of m Boolean expressions in the variables $x_1, \ldots, x_n$ with the restriction that every expression involves at most 3 of these n variables.

Assume we choose a truth assignment u uniformly at random from $\{0, 1\}^n$. Denote then by $\Pr[\phi_i]$ the probability that u satisfies ϕ_i.

(a) Show that $\Pr[\phi_i]$ can be calculated in time polynomial in the length of ϕ_i.

(b) Let N be the random variable which counts the number of expressions ϕ_i satisfied by the random assignment u. Show that

$$\mathbb{E}[N] = \sum_{i=1}^{m} \Pr[\phi_i]\,.$$

(c) We write $\mathbb{E}[N \mid u_1 = 0]$ for the expected number of expressions satisfied by a random assignment u which assigns 0 to x_1. Similarly, define $\mathbb{E}[N \mid u_1 = 1]$. Show:

$$\mathbb{E}[N] = \frac{1}{2} \cdot (\mathbb{E}[N \mid u_1 = 0] + \mathbb{E}[N \mid u_1 = 1]).$$

(d) Show that there is always a value b s.t. $\mathbb{E}[N \mid u_1 = b] \geq \mathbb{E}[N]$.

(e) Give now a polynomial-time algorithms which computes an assignment which satisfies at least $\mathbb{E}[N]$ expressions of Φ.

Exercise 9.3

The decision version of the *traveling salesman problem* (short TSP) is defined as follows:

> Given distances $d_{ij} \geq 0$ between n cities and a bound $B \geq 0$, decide if there is a tour of the cities of length at most B.

We denote the corresponding decision problem by TSP_D, i.e.,

$$\langle d_{1,1}, \ldots, d_{n,n}, B \rangle \in \text{TSP}_D \text{ iff there is TSP-tour w.r.t. } (d_{ij}) \text{ of length at most } B.$$

A *Hamilton path* in an undirected graph $G = (V, E)$ is a path in G which visits every node exactly once. The corresponding decision problem HamiltonPath_D is known to be **NP**-complete.

(a) Show that the following is polynomial-time reduction from HP_D to TSP_D:

Given $G = (V, E)$ with $n = |V|$ and assume that $V = \{1, 2, \ldots, n\}$. Set $d_{i,j} := 1$ if the nodes i and j are connected by some edge, otherwise $d_{i,j} := n + 1$. Further, set $B := 2n$.

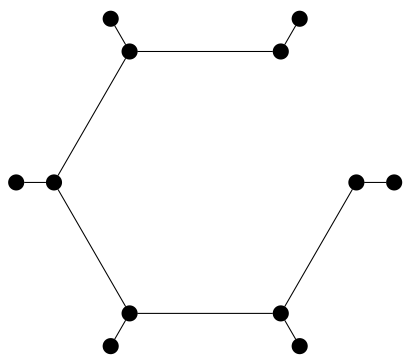
(b) We call a TSP-instance $\langle d_{1,1}, \ldots, d_{n,n}, B \rangle$ *metric* if $d_{ij} \leq d_{ik} + d_{kj}$ holds for all i, j, k.

- Give an example of a graph G where the TSP-instance produced by the reduction above is not metric.
- Modify the reduction such that it always yields a metric TSP-instance.

(c) The following algorithm for approximating the optimal solution of a metric TSP is by Christofides:

First, compute a minimal spanning tree $T = (V_T, E_T)$ of the complete graph K_n with distance matrix (d_{ij}). Let O be the nodes of T which have odd degree (w.r.t. T!). Consider now the complete graph K_O consisting only of these nodes O, and calculate a minimal matching M for it, i.e., find a subset M of the edges of K_O s.t. every node is connected to exactly one other node (no loops) and the total weight of M, i.e., $\sum_{(i,j)\in M} d_{ij}$, is minimal. Add now the edges M to T yielding a multigraph G, i.e., assume that the original edges of T are colored black, while those of M are colored red. Still, the weight of an edge (i, j) in G is d_{ij} independent of its color. Calculate a Eulerian walk of G, i.e., a path of G which uses every edge, both black and red, of G exactly once. The approximation is then the tour embedded in the Eulerian walk.

- Convince yourself that every step of the algorithm by Christofides can be implemented in polynomial time (look it up on the Internet).
- Apply both the algorithm shown in class and the algorithm by Christofides to the following example:



The coordinates of the inner nodes are $R \cdot (\cos \frac{k \cdot 2\pi}{n}, \sin \frac{k \cdot 2\pi}{n})$, for the outer nodes $(R + c) \cdot (\cos \frac{k \cdot 2\pi}{n}, \sin \frac{k \cdot 2\pi}{n})$ where $n = 6$, $k = 1, 2, \ldots, 6$, $R = 2cm$, $c = 0.5cm$. Distances are given by the Euclidean norm if there is an edge, otherwise ∞.

- Try to show that Christofides' algorithm always yields a tour which is at most 50% longer than the optimal tour for a metric TSP-instance.