Technische Universität München (I7) **Summer term 2010**
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Complexity Theory – Homework 4

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Exercise 4.1

An *optimization problem* consists of the following:

- PI: the set of problem instances.
- SC: the set of solution candidates.
- $R \subseteq \text{PI} \times \text{SC}$: the binary relation such that $(x, y) \in R$ iff y is a solution of x.

  We assume that for every $x \in \text{PI}$ there is *always* some (perhaps trivial) solution $y \in \text{SC}$ with $(x, y) \in R$.
- v: the valuation function which maps every pair $(x, y) \in \text{PI} \times \text{SC}$ to an integer $v(x, y)$.

We are then interested in calculating the optimal values

$$\max_{R;v}(x) = \max\{v(x, y) \mid (x, y) \in R\} \text{ resp. } \min_{R;v}(x) := \min\{v(x, y) \mid (x, y) \in R\}.$$

Note that a minimization problem becomes a maximization problem by changing the sign of v. We therefore focus on maximization in the following. The decision problem associated with $\max_{R;v}$ is then

$$\text{Max}^D_{R;v} := \{(x, k) \in \text{PI} \times \mathbb{Z} \mid \exists y : (x, y) \in R \wedge v(x, y) \geq k\}.$$

Example: Let PI be the set of undirected graphs, SC the set of sets of nodes, and let $(x, y) \in R$ iff y is an independet set of x. With $v(x, y)$ the number of nodes in y, we have INDSET $= \text{Max}^D_{R;v}$.

We assume some reasonable (i.e., not unary) encoding of PI and SC and that

- PI, SC, and R are decidable in polynomial time.
- v is computable in polynomial time.
- there is a polynomial p with $|y| \leq p(|x|)$ for every $(x, y) \in R$ (with $|\cdot|$ the length of the encoding).

Remark: Check that these assumptions, in particular the last one, are satisfied by INDSET.

(a) MAXSAT is defined as follows: given a CNF formula ϕ and an integer k, decide if there is a truth assignment which satisfies at least k clauses of ϕ.

Show that MAXSAT is an optimization problem as defined above.

(b) Show that $\text{Max}^D_{R;v}$ is in **NP**.

(c) Show that there is a polynomial r such that $|v(x, y)| \leq r(|x|)$ for all $(x, y) \in R$.

Conclude that in time polynomial in $|x|$ one can compute $b(x), B(x)$ such that $b(x) \leq v(x, y) \leq B(x)$ for all $(x, y) \in R$.

(d) Show that $\max_{R;v}(x)$ can be calculated using binary search and a polynomial number of calls to $\text{Max}^D_{R;v}$.

We might also be interested in maximal solutions, i.e.,

$$\text{Max}_{R;v} := \{(x, z) \in R \mid \forall y : (x, y) \in R \Rightarrow v(x, y) \leq v(x, z)\}$$

(e) Show that $\overline{\text{Max}_{R;v}} \leq_p \text{Max}^D_{R;v}$.

Conclude that $\text{Max}_{R;v} \in \text{co}\mathbf{NP}$ and even $\text{Max}_{R;v} \in \mathbf{NP} \cap \text{co}\mathbf{NP}$ if $\text{Max}^D_{R;v} \in \mathbf{NP} \cap \text{co}\mathbf{NP}$.

We say that $\text{Max}^D_{R;v}$ is *dual* to $\text{Min}^D_{S;w}$ if

$$\text{Max}^D_{R;v} \leq_p \overline{\text{Min}^D_{S;w}} \text{ and } \text{Min}^D_{S;w} \leq_p \overline{\text{Max}^D_{R;v}}.$$

(f) Assume that $\text{Max}^D_{R;v}$ is dual to some minimization problem $\text{Min}^D_{S;w}$. Show that $\text{Max}^D_{R;v} \in \mathbf{NP} \cap \text{co}\mathbf{NP}$.

*(g) Assume that $\text{Max}_{R;v}$ is in $\mathbf{NP} \cap \text{co}\mathbf{NP}$. Show that $\text{Max}^D_{R;v}$ is then dual to some $\text{Min}^D_{S;w}$.

Exercise 4.2

(a) Assume $A \leq_p B$. Show that then also $\overline{A} \leq_p \overline{B}$.

(b) A class $\mathbf{C}$ of languages is *closed under* $\leq_p$ if whenever $A \leq_p B \land B \in \mathbf{C}$, then also $A \in \mathbf{C}$.

Show that if $\mathbf{C}$ is closed under $\leq_p$, then so is $\mathrm{co}\mathbf{C} := \{\overline{L} \mid L \in \mathbf{C}\}$.

(c) Show that co**NP** is closed under intersection and union of languages.

Exercise 4.3

(a) Assume that **P**=**NP**. Show that then **EXP**=**NEXP**.

Remark: Assume that L is decided by some TM running in time $T(n)$ with $T(n)$ time-constructible and $T(n) \in \mathcal{O}(2^{n^c})$ for some $c \geq 1$. Show that then

$$L_{\text{pad}} := \{x10^{T(|x|)}1 \mid x \in L\} \in \mathbf{NP}.$$

*(b) Show that also **EXP**=**NEXP** if only *every unary* **NP**-language is also in **P**.

Remark: For $x \in \{0,1\}^*$ let $\langle x \rangle$ be the natural number represented by x assuming lsbf. Given a language L which is decided in time $T(n)$ (with $T(n)$ time-constructable) show that

$$L_{\text{upad}} = \{1^{\langle x10^{|T(n)|}1 \rangle} \mid x \in L\} \in \mathbf{NP}$$

with $|T(n)|$ ($\approx \lceil \log T(n) \rceil$) the length of the lsbf representation of $T(n)$.