Technische Universität München
Winter term 2018/19
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Automata and Formal Languages — Retake Exam

- You have 120 minutes to complete the exam.
- Answers must be written in a separate booklet. Do not answer on the exam.
- Please let us know if you need more paper.
- Write your name and Matrikelnummer on every sheet.
- Write with a non-erasable pen. Do not use red or green.
- You are not allowed to use auxiliary means other than pen and paper.
- You can obtain 40 points. You need 17 points to pass.

Question 1 $\quad$ **((1 + 1 + 1 + 1) + (2 + 1 + 1+ 1) = 9 points)**

1. Prove: Every regular language can be recognised by an ϵ-NFA with a single initial state and a single final state.
2. Disprove: Every ω-regular language can be recognised by an NBA with a single initial state and a single accepting state.
3. Prove or disprove: For every regular language L, there exists a *planar* ϵ-NFA N such that $L = L(N)$. Recall that a graph is planar if it can be drawn without edge crossings. Note that self-loops are irrelevant for planarity.
4. Let $\Sigma = \{a, b\}$ be a finite alphabet. Give a sentence of MSO(Σ) for the language $L = a^*b^*$.

Let D be a deterministic automaton and N be a nondeterministic automaton. Let $L(D)$ denote the language of D when we interpret D as a DFA and $L_\omega(D)$ when we interpret D as a DBA. Analogously, let $L(N)$ denote the language of N when we interpret N as a NFA and $L_\omega(N)$ when we interpret N as a NBA.

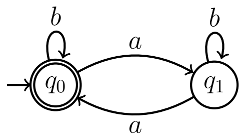
5. Prove: $L(D)$ is infinite if and only if $L_\omega(D)$ is non-empty.
6. Disprove: $L(N)$ is infinite if and only if $L_\omega(N)$ is non-empty.
7. Prove or disprove: The LTL formulas $(\mathbf{G}p)\ \mathbf{U}\ (\mathbf{G}q)$ and $\mathbf{G}(p\ \mathbf{U}\ q)$ are equivalent.
8. Give an ω-regular expression over the alphabet $\Sigma = 2^{\{a,b\}}$ for the formula $\mathbf{FG}a \wedge \mathbf{GF}b$.

Question 2 (2 + 3 = 5 points)
The *perfect shuffle* of two languages $L, L' \in \Sigma^*$ is defined as:

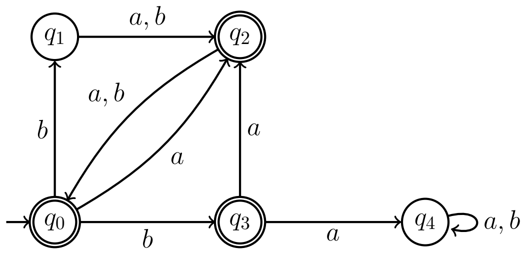
$$L \,\widetilde{\|}\, L' := \{ w = a_1 b_1 \cdots a_n b_n \in \Sigma^* : \; a_1 \cdots a_n \in L \wedge b_1 \cdots b_n \in L' \}$$

Notice that $a_1 \cdots a_n$ and $b_1 \cdots b_n$ have to have the same length. For example, if $L_1 = \{\epsilon, a, bb\}$ and $L_2 = \{\epsilon, d, cc, eee\}$ then $L_1 \,\widetilde{\|}\, L_2 = \{\epsilon, ad, bcbc\}$.

1. Give a construction that takes two DFAs A and B as input, and returns a DFA accepting $L(A) \,\widetilde{\|}\, L(B)$.
2. Apply your construction to two copies of the following DFA D and give an interpretation of the states of the resulting DFA:



Question 3 (4 points)
Use the algorithm *UnivNFA* to test whether the following NFA is universal.



Question 4 (3 + 3 = 6 points)
Consider languages over the alphabet $\Sigma = \{0, 1\}$ with fixed length of 3.

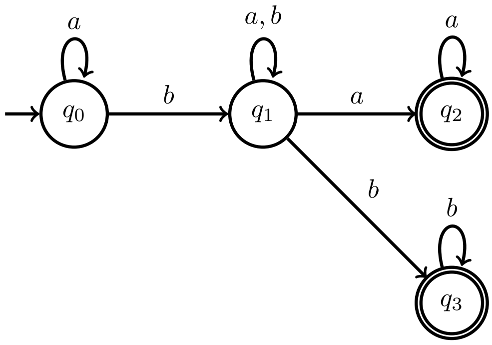
1. Let $L \subseteq \Sigma^3$ be the set of all prime numbers in the range $[0, 7]$, i.e. $\{2, 3, 5, 7\}$, represented in binary using the msbf encoding. Construct the corresponding fragment of the master automaton for L.
2. Decide whether there exists a language $L \subseteq \Sigma^3$ such that the minimal DFA for this language has at least 11 states. Explain your answer.

Question 5 $(3 + 2 + 2 + 4 + 1 = 12$ points)

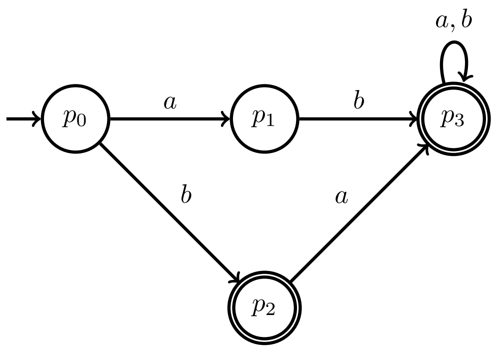
A Büchi automaton is *very-weak* if the only cycles in the automaton (when the transition relation is viewed as a directed graph) are self-loops.

Consider now the following two very-weak NBAs N_1 and N_2:

N_1 :



N_2 :



1. Show that very-weak NBAs are closed under union and intersection by giving constructions for union and intersection. Argue that these constructions yield very-weak NBAs and the correct languages.
2. Apply the intersection construction you defined in (1) to N_1 and N_2.
3. Construct a very-weak NBA for the LTL formula $\varphi = (a\ \mathbf{U}\ \mathbf{X}b) \vee \mathbf{F}c$.
4. By LTL($\mathbf{F}$,$\mathbf{X}$) we denote the fragment of LTL with the restricted syntax of:

$$\varphi \coloneqq a \mid \neg a \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \mathbf{F}\varphi \mid \mathbf{X}\varphi$$

   Sketch a (recursive) procedure that translates LTL($\mathbf{F}$,$\mathbf{X}$) to very-weak NBAs.
5. Give an LTL formula that cannot be recognised by very-weak NBAs.

Question 6 $(2 + 2 = 4$ points)

Let $L \subseteq \Sigma^*$ be a language of finite words. We define the *limit ω-language* $\overrightarrow{L}$ of a given language L as follows:

$$w \in \overrightarrow{L} \iff \text{infinitely many prefixes of } w \text{ are in } L$$

For example, if $L = b + (ab)^*$ then $\overrightarrow{L} = (ab)^\omega$.

1. Give an NFA N such that the limit of its language and the language of N viewed as a Büchi automaton differ.
2. Prove that for any DFA D, the limit of its language and the language of D viewed as a Büchi automaton coincide.

Solution 1 $((1+1+1+1) + (2+1+1+1) = 9$ **points)**

1. True. Consequence of the translation from *RE*-NFA to NFA translation. Alternatively two new states (one initial, one accepting) can be added and be connected to the appropriate states using ϵ-transitions.

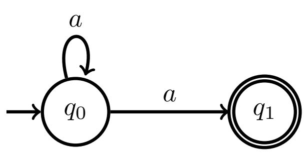
2. False. Consider the NBAs for $a^\omega + b^\omega$. If there is single accepting state, also $a^n b^\omega$ (for some n) is accepted.

3. True. The *RE*-NFA with two states and a single transition labelled with r is planar. Each application of the rules to obtain a ϵ-NFA preserves planarity and thus we obtain planar ϵ-NFA for the regular language $L(r)$.

4. $\forall x.\forall y.(Q_a(x) \wedge Q_b(y)) \rightarrow x < y$

5. If $L(D)$ is infinite, there exists a reachable accepting cycle in the automaton. Thus we can construct an accepting run on the DBA D piece-wise following this cycle. If $L_\omega(D)$ is non-empty, then there exists an accepting lasso in D. This lasso can be followed for a infinite number of times. Thus $L(D)$ is infinite.

6. The following NFA N recognises a^+, but $L_\omega(N)$ is empty.



7. False. Let $w = (\{p\}\{q\})^\omega$. Then $w \not\models (\mathbf{G}p)\ \mathbf{U}\ (\mathbf{G}q)$, but $w \models \mathbf{G}(p\ \mathbf{U}\ q)$.

8. $\Sigma^*(\{a\}^*\{a,b\})^\omega$.

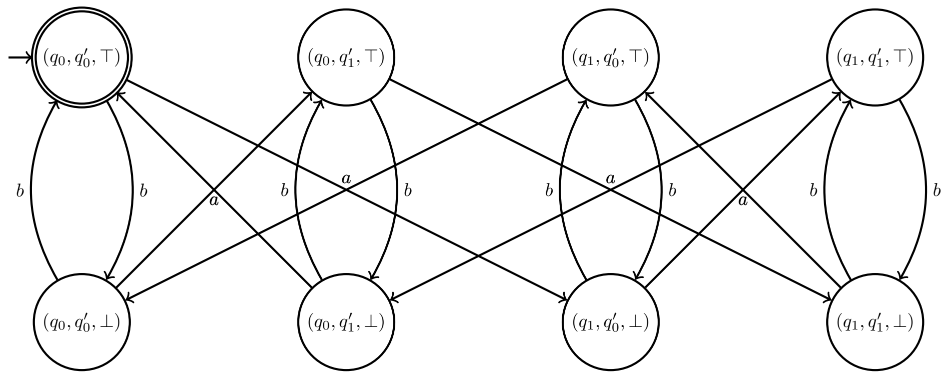
Solution 2 $(2+3 = 5$ **points)**

1. Let $A = (Q, \Sigma, \delta, q_0, F)$ and $B = (Q', \Sigma, \delta', q_0', F')$. Intuitively, we build a DFA C that alternates between reading a letter in A and reading a letter in B. To do so, we build two copies of the product of A and B. Reading a letter a in the first copy simulates reading a in A and then goes to the bottom copy, and vice versa. A word is accepted if it ends up in a state (p, q) of the top copy such that $p \in F$ and $q \in F'$.

   Formally, $C = (Q'', \Sigma, \delta'', q_0'', F'')$ where

   - $Q'' = Q \times Q' \times \{\top, \bot\}$
   - $\delta(p, a) = \begin{cases} (\delta(q,a), q', \bot) & \text{if } p = (q, q', \top) \\ (q, \delta'(q', a), \top) & \text{if } p = (q, q', \bot) \end{cases}$
   - $q_0'' = (q_0, q_0', \top)$
   - $F'' = \{(q, q', \top) : q \in F \wedge q' \in F'\}$

2. Applying the construction to D and D' (a copy of D) yields:



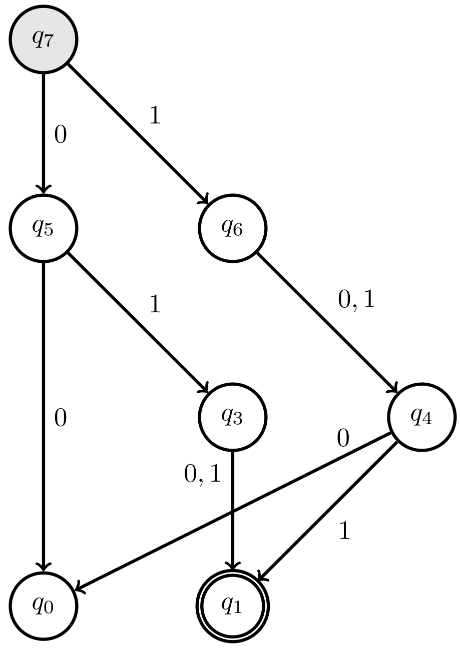
Solution 3 (4 points)

Iter.	$\mathcal{Q}$	$\mathcal{W}$
0	$\emptyset$	$\{\{q_0\}\}$
1	$\{\{q_0\}\}$	$\{\{q_2\}, \{q_1, q_3\}\}$
2	$\{\{q_0\}, \{q_2\}\}$	$\{\{q_1, q_3\}\}$
3	$\{\{q_0\}, \{q_2\}, \{q_1, q_3\}\}$	$\emptyset$

The algorithm returns true, hence the NFA accepts $\{a, b\}^*$.

Solution 4 (3 + 3 = 6 points)

1. The resulting fragment of the Master automaton is:



2. There is no such language. Assume there exists such a language and let be D the corresponding minimal DFA. This DFA is also contained in the master automaton. The first layer has exactly one state, the following layer at most two, and the following at most four. Including the last layer with two states the automaton has at most $1 + 2 + 4 + 2 = 9$ states and which contradicts our assumption.

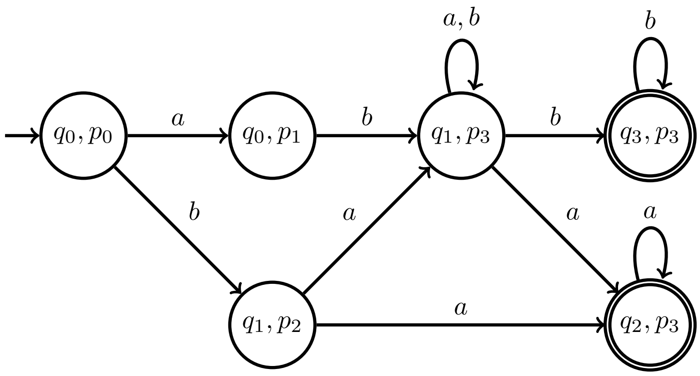
Solution 5 (3 + 2 + 2 + 4 + 1 = 12 points)

1. The union of two very-weak NBAs is constructed by taking the disjoint union of the states, initial states, accepting states, and transitions. On the one hand, the constructed automaton recognises the union of languages, since it can non-deterministically guess the correct initial state. On the other hand, whenever the union automaton accepts there is a corresponding accepting run on one of the (input) NBAs. Furthermore, since we did not change the transition structure, the resulting NBA is also very-weak.

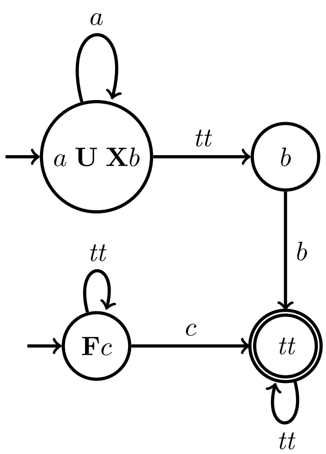
   The intersection construction builds the pairing of both input automata. Notice that the result is again very-weak, since no (non-trivial) cycles are created: To be more precise assume we would have the cycle $(q, p) \to (q', p') \to (q, p)$ and $q \neq q'$ or $p \neq p'$. Then either the run in q or p has to leave the state and to return to it which is not possible since the underlying automata are very-weak. For intersection on general Büchi automata we need to add copies to ensure that both sets of accepting states are visited infinitely often. However, when an NBA is very-weak this is not necessary, since each run eventually gets stuck in a single state and either both components are accepting (mark the pairing-state accepting) or one state is non-accepting (and thus we mark the pairing-state non-accepting).

   Formally let N and N' be two NBAs over the same finite alphabet Σ. Then the intersection automaton is defined as follows: NBA $N'' = (Q \times Q', \Sigma, \delta \times \delta', (q_0, q'_0), F \times F')$. The correctness follows from the correctness of the pairing construction and the previous explanation.

2. Very-weak intersection NBA:



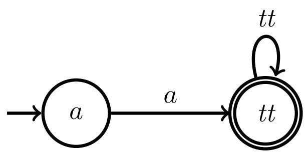
3. Very-weak NBA for φ:



4. An automaton can be built bottom-up using the following rules matching the syntax:

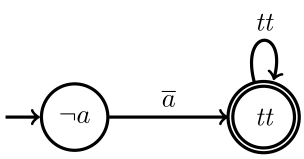
   - $\varphi = a$:

     N_a :



   - $\varphi = \neg a$:

     $N_{\neg a}$:



   - $\varphi = \psi_1 \wedge \psi_2$: Let N_{ψ_1} and N_{ψ_2} be the very-weak NFAs for ψ_1 and ψ_2. We then apply the intersection construction from (1) to obtain $N_{\psi_1 \wedge \psi_2}$.
   - $\varphi = \psi_1 \vee \psi_2$: Let N_{ψ_1} and N_{ψ_2} be the very-weak NFAs for ψ_1 and ψ_2. We then apply the union construction from (1) to obtain $N_{\psi_1 \vee \psi_2}$.
   - $\varphi = \mathbf{X}\psi$: Let $N_\psi = (Q, \Sigma, \delta, Q_0, F)$ be the very-weak NFAs for ψ. The following very-weak NBA $N_{\mathbf{X}\psi}$ then accepts the language $\mathbf{X}\psi$:

$$N_{\mathbf{X}\psi} = (Q \cup \{\mathbf{X}\psi\}, \Sigma, \delta \cup \{(\mathbf{X}\psi, a, q_0) : a \in \Sigma, q_0 \in Q_0\}, \{\mathbf{X}\psi\}, F)$$

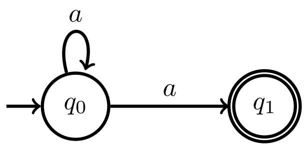
- $\varphi = \mathbf{F}\psi$: Let $N_\psi = (Q, \Sigma, \delta, Q_0, F)$ be the very-weak NFAs for ψ. The following very-weak NBA $N_{\mathbf{F}\psi}$ then accepts the language $\mathbf{F}\psi$:

$$N_{\mathbf{F}\psi} = (Q, \Sigma, \delta \cup \{(q_0, a, q_0) : a \in \Sigma, q_0 \in Q_0\}, \{\mathbf{X}\psi\}, F)$$

5. $\varphi = \mathbf{G}(a \leftrightarrow \mathbf{X}b)$.

Solution 6 $\quad$ (2 + 2 = 4 points)

1. The following NFA N recognises a^+ and the limit of its language is a^ω, but $L_\omega(N)$ is empty.



2. Let D be a DFA. We need to show that $\overrightarrow{L(D)} = L_\omega(D)$ holds.

   Let $w \in L_\omega(D)$ be an accepted word. Then there exists an accepting run r on the automaton D and hence there are infinitely many i's such that $r[i] \in F$. Hence for each i the prefix of length i (denoted w_{0i}) is accepted by the DFA D. As we have seen, there exist infinitely many such prefixes and thus $w \in \overrightarrow{L(D)}$.

   Let $w \notin L_\omega(D)$ be a rejected word. Since the automaton is deterministic, there is a unique run r. Because this run is rejecting, after reading some prefix w' of w the run never visits an accepting state again. Thus there are only finitely many prefixes of w that are accepted by the DFA D and thus $w \notin \overrightarrow{L(D)}$.