Technische Universität München **Winter term 2020/21**
I7
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Automata and Formal Languages — Exercise Sheet 2

Exercise 2.1

Determine the residuals of the following languages:

(a) $(aa + bb)^*$ over $\Sigma = \{a, b\}$,

(b) $(abc)^*$ over $\Sigma = \{a, b, c\}$,

(c) $\{a^n b^n c^n \mid n \geq 0\}$ over $\Sigma = \{a, b, c\}$,

(d) $\{a^n b^{3n} \mid n \geq 0\}$ over $\Sigma = \{a, b\}$.

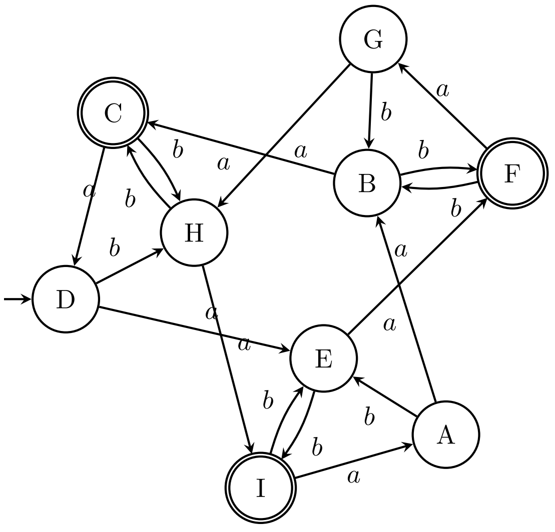
Exercise 2.2

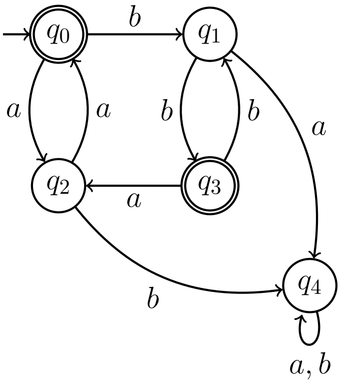
(a) Let $\Sigma = \{0, 1\}$ be an alphabet.
Find a language $L \subseteq \Sigma^*$ that has infinitely many residuals and $|L^w| > 0$ for all $w \in \Sigma^*$.

(b) Let $\Sigma = \{a\}$ be an alphabet.
Find a language $L \subseteq \Sigma^*$, such that $L^w = L^{w'} \implies w = w'$ for all words $w, w' \in \Sigma^*$.
What can you say about the residuals for such a language L? Is such a language regular?

Exercise 2.3

Let A and B be respectively the following DFAs:





(a) Compute the language partitions of A and B.

(b) Construct the quotients of A and B with respect to their language partitions.

(c) Give regular expressions for $L(A)$ and $L(B)$.

Exercise 2.4

Let $\text{msbf}\colon \{0,1\}^* \to \mathbb{N}$ be such that $\text{msbf}(w)$ is the number represented by w in the "most significant bit first" encoding[1]. For example,

$$\text{msbf}(1010) = 10,\ \text{msbf}(100) = 4,\ \text{msbf}(0011) = 3.$$

For every $n \geq 2$, let us define the following language:

$$M_n = \{w \in \{0,1\}^* :\ \text{msbf}(w) \text{ is a multiple of } n\}.$$

(a) Show that M_3 has (exactly) three residuals, i.e. show that $|\{(M_3)^w : w \in \{0,1\}^*\}| = 3$.

(b) Show that M_4 has less than four residuals.

(c) Show that M_p has (exactly) p residuals for every prime number p. You may use the fact that, by Fermat's little theorem, $2^{p-1} \equiv 1 \pmod{p}$.

[Hint: For every $0 \leq i < p$, consider the word u_i such that $|u_i| = p - 1$ and $\text{msbf}(u_i) = i$.]

[1] Recall this type of encoding from Exercise 1.4 from the previous exercise sheet. In contrast to the function MSBF, this one (msbf) maps an encoding to its corresponding natural number.

Solution 2.1

- For $(aa+bb)^*$. We give the residuals as regular expressions: $(aa+bb)^*$ (residual with respect to ε); $a(aa+bb)^*$ (residual with respect to a); $b(aa+bb)^*$ (residual with respect to b); $\emptyset$ (residual with respect to ab). All other residuals are equal to one of these four.

- For $(abc)^*$. We give the residuals as regular expressions: $(abc)^*$ (residual of ε); $bc(abc)^*$ (residual of a); $c(abc)^*$ (residual of ab); $\emptyset$ (residual of b). All other residuals are equal to one of these three.

- For $L = \{a^n b^n c^n \mid n \geq 0\}$: Every prefix of a word of the form $a^n b^n c^n$ has a different residual. For all other words the residual is the empty set. There are infinitely many residuals:
  - $L^\varepsilon = L$,
  - for every $i \geq 1$, we have a residual with respect to a^i, which is $L^{a^i} = \{a^{n-i} b^n c^n \mid n \geq i\}$,
  - for every $n \geq i \geq 1$ we have a residual with respect to $a^n b^i$, which is $L^{a^n b^i} = \{b^{n-i} c^n\}$,
  - for every $n \geq i \geq 1$ we have a residual with respect to $a^n b^n c^i$, which is $L^{a^n b^n c^i} = \{c^{n-i}\}$,
  - $L^b = \emptyset$.

- Similarly for $L = \{a^n b^{3n} \mid n \geq 0\}$, every prefix of a word of the form $a^n b^{3n}$ has a different residual:
  - $L^\varepsilon = L$,
  - for every $i \geq 1$, we have a residual with respect to a^i, which is $L^{a^i} = \{a^{n-i} b^{3n} \mid n \geq i\}$,
  - for every $3n \geq i \geq 1$ we have a residual with respect to $a^n b^i$, which is $L^{a^n b^i} = \{b^{3n-i}\}$,
  - $L^b = \emptyset$.

Solution 2.2

(a) $L = \{ww \mid w \in \Sigma^*\}$. First we prove that L has infinitely many residuals by showing that for each pair of words of the infinite set $\{0^i 1 \mid i \geq 0\}$ the corresponding residuals are not equal. Let $u = 0^i 1, v = 0^j 1 \in \Sigma^*$ two words with $i < j$. Then $L^u \neq L^v$ since $u \in L^u$, but $u \notin L^v$. For the second half consider some arbitrary word w. Then $w \in L^w$, which shows the statement.

(b) We observe that for all languages satisfying that property L^w has to be non-empty for all w and thus also infinite. Furthermore all these languages are not regular, since there are infinitely many residuals.

$L = \{a^{2^n} \mid n \geq 0\}$. Let a^i and a^j two distinct words. W.l.o.g. we assume $i < j$. Let now d_i and d_j denote the distance from i and j to resp. closest power of 2. If $d_i < d_j$ holds, we are immediately done since $a^{d_i} \in L^{a^i}$ and $a^{d_i} \notin L^{a^j}$. $d_i > d_j$ is analogous. Thus assume $d_i = d_j$. Let us then define d_i' and d_j' denote the distance from i and j to resp. second closest power of 2. These have to be unequal, since the gaps between the powers of 2 are strictly increasing and we can repeat the argument from before.

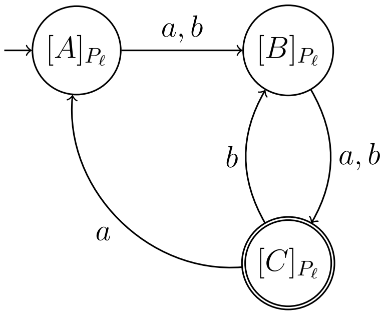
Solution 2.3

A) (a)

Iter.	**Block to split**	**Splitter**	**New partition**
0	—	—	$\{A,B,D,E,G,H\}, \{C,F,I\}$
1	$\{A,B,D,E,G,H\}$	$(b, \{A,B,D,E,G,H\})$	$\{A,D,G\}, \{B,E,H\}, \{C,F,I\}$
2	none, partition is stable	—	—

The language partition is $P_\ell = \{\{A,D,G\}, \{B,E,H\}, \{C,F,I\}\}$.

(b) The minimal automaton is given below:



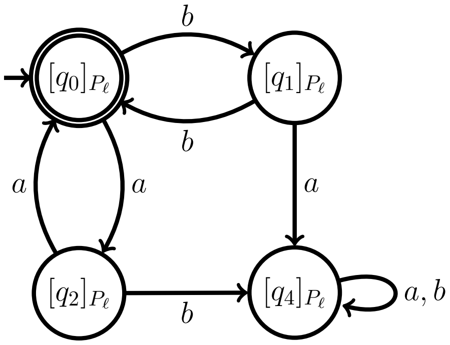
(c) $\Sigma\Sigma(a\Sigma\Sigma + b\Sigma)^*$.

B) (a)

Iter.	Block to split	Splitter	New partition
0	—	—	$\{q_0, q_3\}, \{q_1, q_2, q_4\}$
1	$\{q_1, q_2, q_4\}$	$(b, \{q_1, q_2, q_4\})$	$\{q_0, q_3\}, \{q_1\}, \{q_2, q_4\}$
2	$\{q_2, q_4\}$	$(a, \{q_0, q_3\})$	$\{q_0, q_3\}, \{q_1\}, \{q_2\}, \{q_4\}$
3	none, partition is stable	—	—

The language partition is $P_\ell = \{\{q_0, q_3\}, \{q_1\}, \{q_2\}, \{q_4\}\}$.

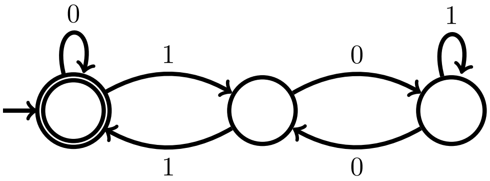
(b)



(c) $(aa + bb)^*$ or $((aa)^*(bb)^*)^*$.

Solution 2.4

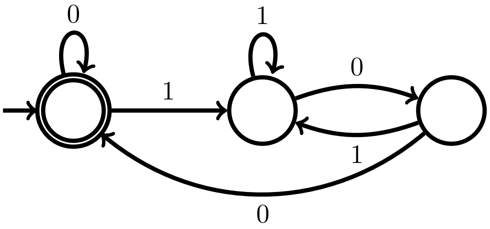
(a) The following DFA accepts M_3. The states represent congruence classes w.r.t. the modulo 3 relation.



As this DFA has three states, therefore M_3 has at most three residuals. We claim that M_3 has *at least* three residuals. To prove this claim, it suffices to show that the ε-residual (M_3^ε), 1-residual (M_3^1) and 10-residual (M_3^{10}) of M_3 are distinct.

- Since $\varepsilon \cdot \varepsilon \in M_3$ and $1 \cdot \varepsilon \notin M_3$, we know that $\varepsilon \in M_3^\varepsilon$ but $\varepsilon \notin M_3^1$, and thus $M_3^\varepsilon \neq M_3^1$.
- Since $\varepsilon \cdot \varepsilon \in M_3$ and $10 \cdot \varepsilon \notin M_3$, we know that $\varepsilon \in M_3^\varepsilon$ but $\varepsilon \notin M_3^{10}$, and thus $M_3^\varepsilon \neq M_3^{10}$.
- Since $1 \cdot 1 \in M_3$ and $10 \cdot 1 \notin M_3$, we know that $1 \in M_3^1$ but $1 \notin M_3^{10}$, and thus $M_3^1 \neq M_3^{10}$.

(b) The following DFA accepts M_4. You can obtain in two steps: (i) construct a DFA with four states that accepts M_4, where each state represents a congruence class w.r.t. the modulo 4 relation, (ii) minimize it.



As it has three states, M_4 has at most three residuals. □

(c) A DFA accepting M_p can be defined as $A_p = (Q_p, \{0, 1\}, \delta_p, 0, \{0\})$ where

$$\begin{aligned} Q_p &= \{0, 1, \ldots, p-1\}, \\ \delta_p(q, b) &= (2q + b) \bmod p \text{ for every } q \in Q_p \text{ and } b \in \{0, 1\}. \end{aligned}$$

As this DFA has p states, then M_p has at most p residuals. It remains to show that M_p has at least p residuals. For every $0 \leq i < p$, let u_i be the word such that $|u_i| = p - 1$ and $\mathrm{msbf}(u_i) = i$. Note that u_i exists since the smallest encoding of i has at most $p - 1$ bits, and it can be extended to length $p - 1$ by padding with zeros on the left. Let us show that the u_i-residual and u_j-residual of M_p are distinct for every $0 \leq i, j < p$ such that $i \neq j$. Let $0 \leq k < p$, and let $\ell = (p - i) \bmod p$. We have:

$$\begin{aligned} \mathrm{msbf}(u_k u_\ell) &= 2^{|u_\ell|} \cdot \mathrm{msbf}(u_k) + \mathrm{msbf}(u_\ell) \\ &= 2^{p-1} \cdot k + ((p - i) \bmod p) \\ &\equiv k + ((p - i) \bmod p) && \text{(by Fermat's little theorem)} \\ &\equiv k + p - i \\ &\equiv k - i. \end{aligned}$$

Let $0 \leq i, j < p$ be such that $i \neq j$. We have $u_i u_\ell \in M_p$ since $\mathrm{msbf}(u_i u_\ell) \equiv i - i \equiv 0$, but we have $u_j u_\ell \notin M_p$ since $\mathrm{msbf}(u_j u_\ell) \equiv j - i \not\equiv 0$. Therefore, the u_i-residual and u_j-residual of M_p are distinct. □