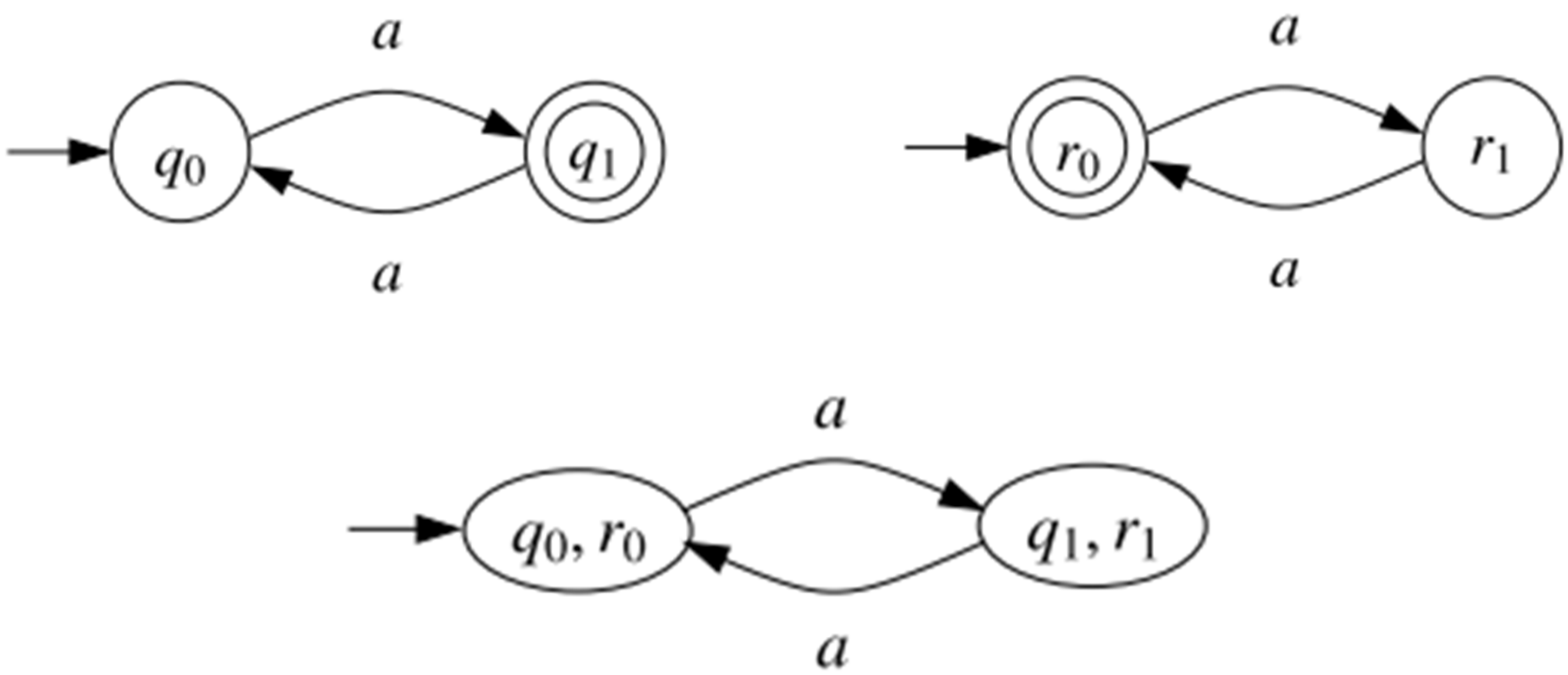
Implementing boolean operations for Büchi automata

Intersection of NBAs

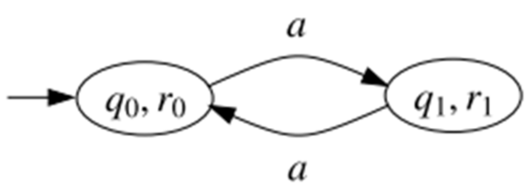
- The algorithm for NFAs does not work ...

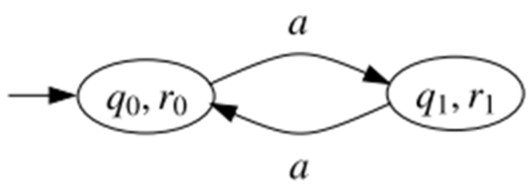


Solution

Apply the same idea as in the conversion NGA ⇒ NBA

1. Take two copies of the pairing $[A_1, A_2]$.

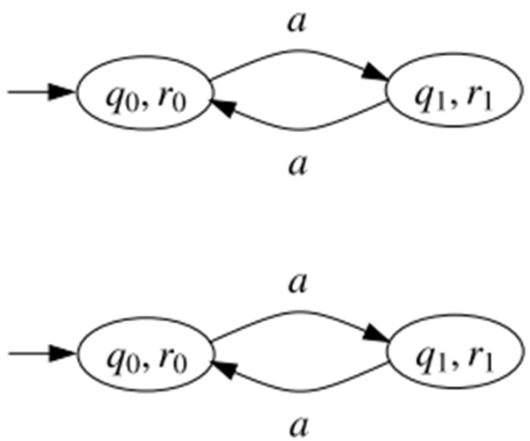




Solution

Apply the same idea as in the conversion NGA ⇒ NBA

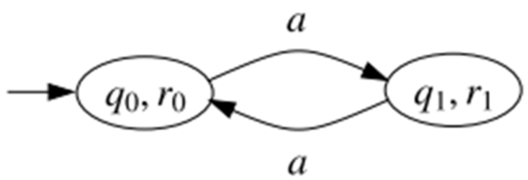
1. Take two copies of the pairing $[A_1, A_2]$.
2. Redirect transitions of the first copy leaving F_1 to the second copy.

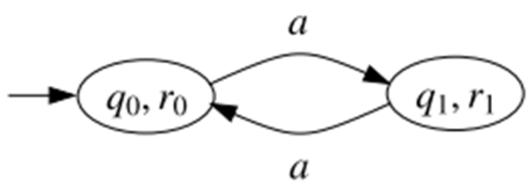


Solution

Apply the same idea as in the conversion NGA ⇒ NBA

1. Take two copies of the pairing $[A_1, A_2]$.
2. Redirect transitions of the first copy leaving F_1 to the second copy.
3. Redirect transitions of the second copy leaving F_2 to the second copy.

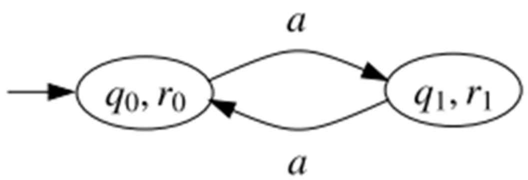


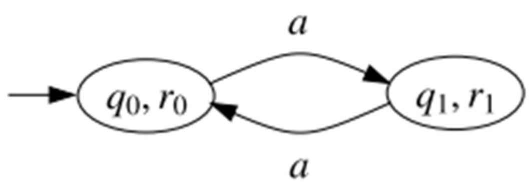


Solution

Apply the same idea as in the conversion NGA ⇒ NBA

1. Take two copies of the pairing $[A_1, A_2]$.
2. Redirect transitions of the first copy leaving F_1 to the second copy.
3. Redirect transitions of the second copy leaving F_2 to the second copy.
4. Set F to the set F_1 in the first copy.





IntersNBA(A_1, A_2)
Input: NBAs $A_1 = (Q_1, \Sigma, \delta_1, q_{01}, F_1)$, $A_2 = (Q_2, \Sigma, \delta_2, q_{02}, F_2)$
Output: NBA $A_1 \cap_\omega A_2 = (Q, \Sigma, \delta, q_0, F)$ with $L_\omega(A_1 \cap_\omega A_2) = L_\omega(A_1) \cap L_\omega(A_2)$

$Q, \delta, F \leftarrow \emptyset$
$q_0 \leftarrow [q_{01}, q_{02}, 1]$
$W \leftarrow \{\ [q_{01}, q_{02}, 1]\ \}$
while $W \neq \emptyset$ **do**
    pick $[q_1, q_2, i]$ **from** W
    add $[q_1, q_2, i]$ **to** Q'
    if $q_1 \in F_1$ **and** $i = 1$ **then add** $[q_1, q_2, 1]$ **to** F'
    for all $a \in \Sigma$ **do**
        for all $q'_1 \in \delta_1(q_1, a), q'_2 \in \delta(q_2, a)$ **do**
            if $i = 1$ **and** $q_1 \notin F_1$ **then**
                add $([q_1, q_2, 1], a, [q'_1, q'_2, 1])$ **to** δ
                if $[q'_1, q'_2, 1] \notin Q'$ **then add** $[q'_1, q'_2, 1]$ **to** W
            if $i = 1$ **and** $q_1 \in F_1$ **then**
                add $([q_1, q_2, 1], a, [q'_1, q'_2, 2])$ **to** δ
                if $[q'_1, q'_2, 2] \notin Q'$ **then add** $[q'_1, q'_2, 2]$ **to** W
            if $i = 2$ **and** $q_2 \notin F_2$ **then**
                add $([q_1, q_2, 2], a, [q'_1, q'_2, 2])$ **to** δ
                if $[q'_1, q'_2, 2] \notin Q'$ **then add** $[q'_1, q'_2, 2]$ **to** W
            if $i = 2$ **and** $q_2 \in F_2$ **then**
                add $([q_1, q_2, 2], a, [q'_1, q'_2, 1])$ **to** δ
                if $[q'_1, q'_2, 1] \notin Q'$ **then add** $[q'_1, q'_2, 1]$ **to** W
return $(Q, \Sigma, \delta, q_0, F)$

Special cases/improvements

- If all states of at least one of A_1 and A_2 are accepting, the algorithm for NFAs works.
- Intersection of NBAs $A_1, A_2, \ldots, A_k$
  - Do NOT apply the algorithm for two NBAs $(k-1)$ times.
  - Proceed instead as in the translation

    NGA $\Rightarrow$ NBA: take k copies of $[A_1, A_2, \ldots, A_k]$

    ($k n_1 \ldots n_k$ states instead of $2^k n_1 \ldots n_k$)

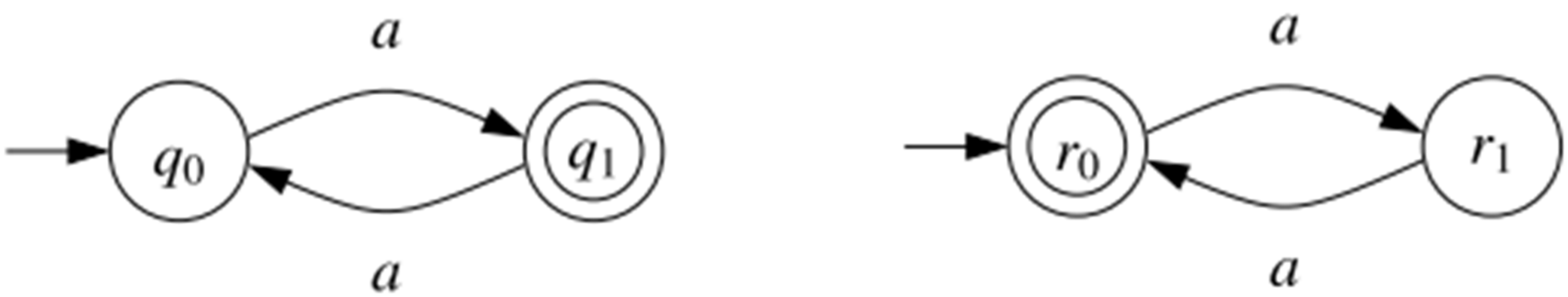
Complement

- Main result proved by Büchi: NBAs are closed under complement.
- Many later improvements in recent years.
- Construction radically different from the one for NFAs.

Problems

- The powerset construction does not work.

- Exchanging final and non-final states in DBAs also fails.

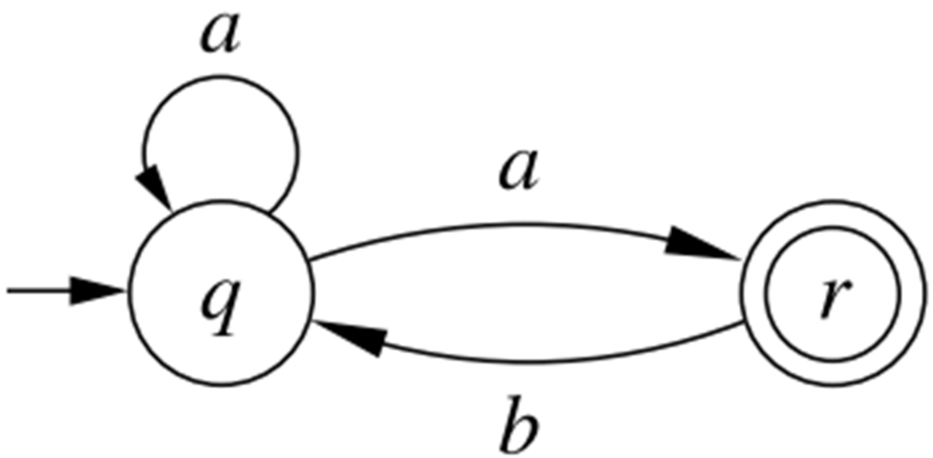


Solution

- Extend the idea used to determinize co-Büchi automata with a new component.
- Recall: a NBA accepts a word w iff some path of $dag(w)$ visits final states infinitely often.
- Goal: given NBA A, construct NBA $\bar{A}$ such that:

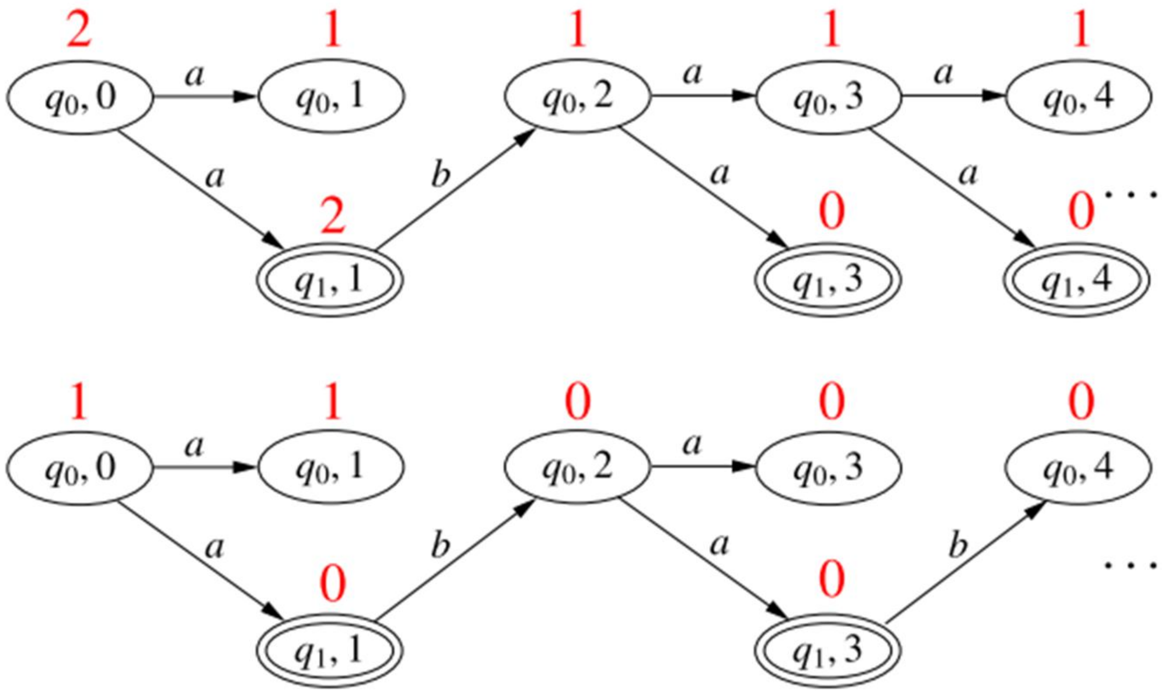
A rejects w
iff
no path of $dag(w)$ visits accepting states of A i.o.
iff
some run of $\bar{A}$ visits accepting states of $\bar{A}$ i.o.
iff
$\bar{A}$ accepts w

Running example



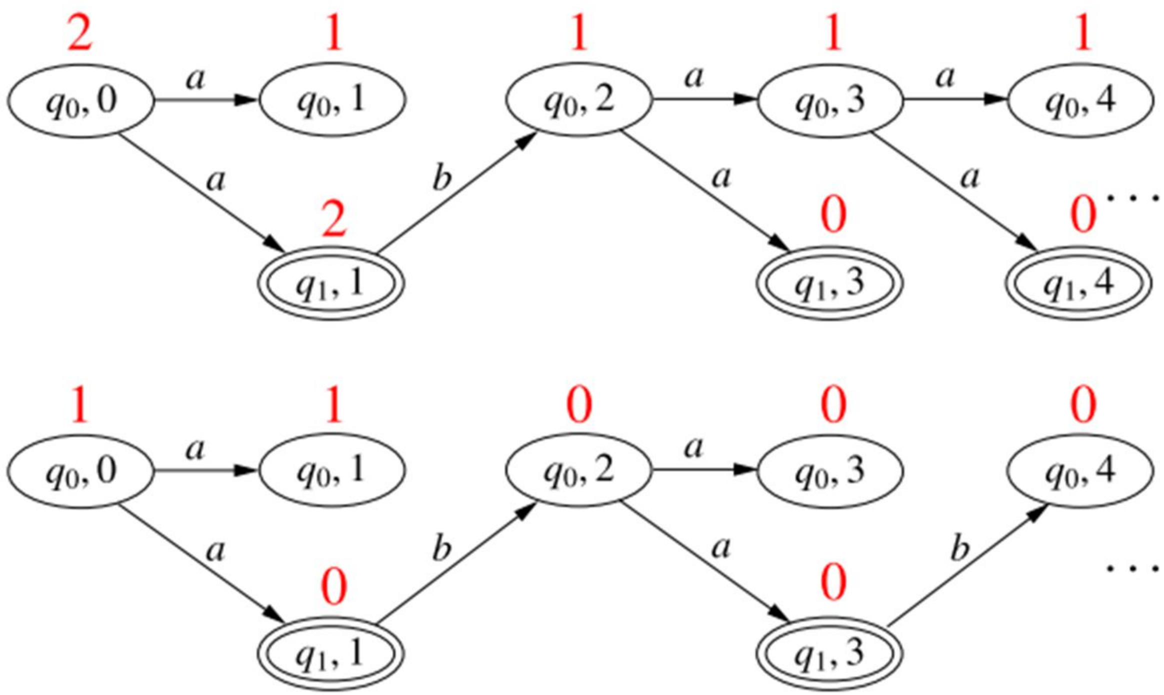
Rankings

- Mappings that associate to every node of $dag(w)$ a rank (a natural number) such that
  - ranks never increase along a path, and
  - ranks of accepting nodes are even.



Odd rankings

- A ranking is odd if every infinite path of $dag(w)$ visits nodes of odd rank i.o.



Prop.: no path of $dag(w)$ visits accepting states of A i.o.

iff

$dag(w)$ has an odd ranking

Proof: Ranks along infinite paths eventually reach a stable rank.

(←): The stable rank of every path is odd. Since accepting nodes have even rank, no path visits accepting nodes i.o.

(→): We construct a ranking satisfying the conditions.

Give each accepting node $\langle q, l \rangle$ rank $2k$, where k is the maximal number of accepting nodes in a path starting at $\langle q, l \rangle$.

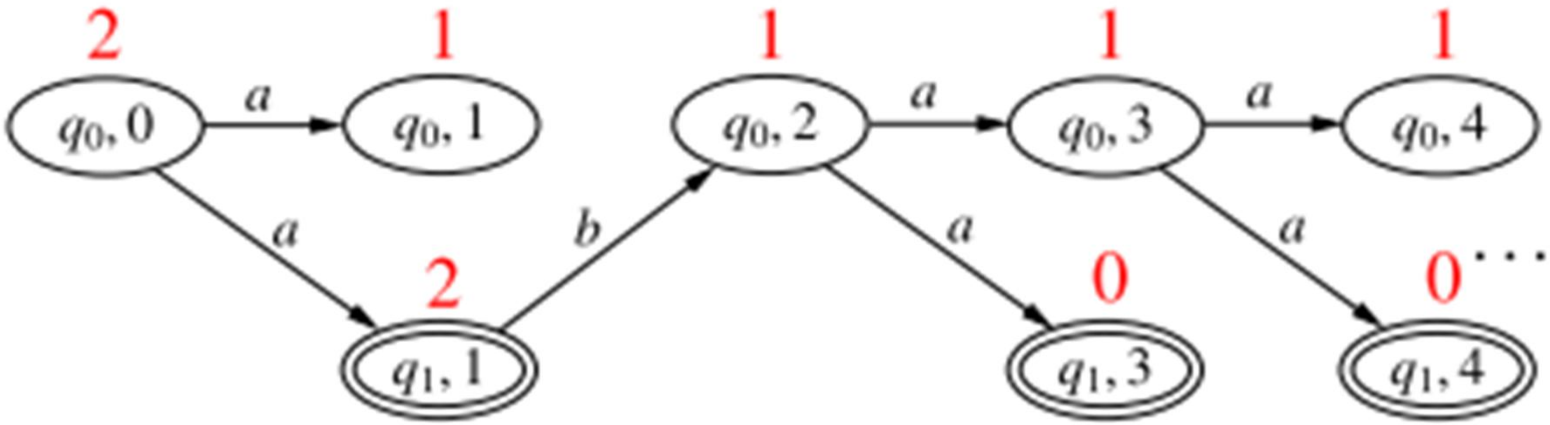
Give a non-accepting node $\langle q, l \rangle$ rank $2k + 1$, where 2k is the maximal even rank among its descendants.

- Goal:

A rejects w
iff
$dag(w)$ has an odd ranking
iff
some run of $\bar{A}$ visits accepting states of $\bar{A}$ i.o.
iff
$\bar{A}$ accepts w

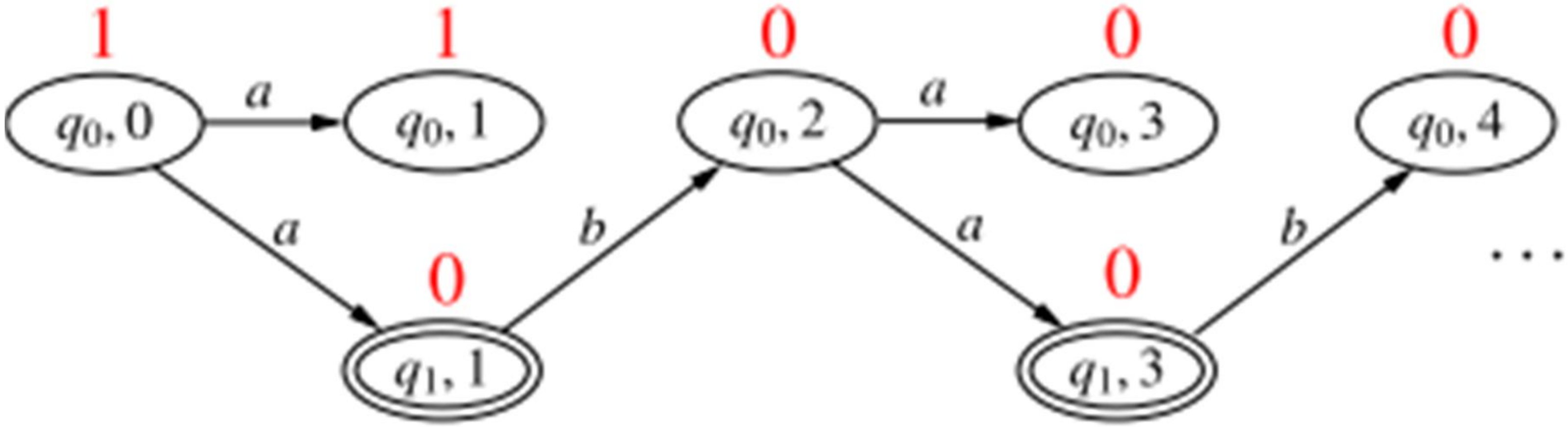
- Idea: design $\bar{A}$ so that
  - its runs on w are the rankings of $dag(w)$, and
  - its acceptings runs on w are the odd rankings of $dag(w)$.

Representing rankings



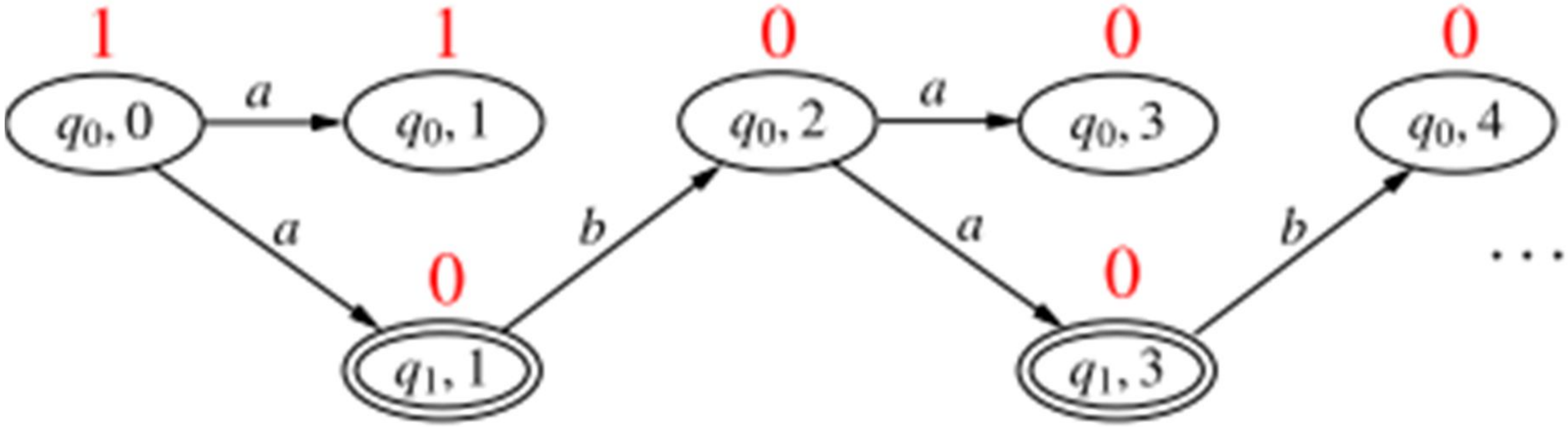
$$\begin{bmatrix}2\\ \bot\end{bmatrix} \xrightarrow{a} \begin{bmatrix}1\\ 2\end{bmatrix} \xrightarrow{b} \begin{bmatrix}1\\ \bot\end{bmatrix} \xrightarrow{a} \begin{bmatrix}1\\ 0\end{bmatrix} \xrightarrow{a} \begin{bmatrix}1\\ 0\end{bmatrix} \quad \cdots$$

Representing rankings



$$\begin{bmatrix}1\\ \bot\end{bmatrix} \xrightarrow{a} \begin{bmatrix}1\\ 0\end{bmatrix} \xrightarrow{b} \begin{bmatrix}0\\ \bot\end{bmatrix} \xrightarrow{a} \begin{bmatrix}0\\ 0\end{bmatrix} \xrightarrow{b} \begin{bmatrix}0\\ \bot\end{bmatrix} \cdots$$

Representing rankings



$$\begin{bmatrix}1\\ \bot\end{bmatrix} \xrightarrow{a} \begin{bmatrix}1\\ 0\end{bmatrix} \xrightarrow{b} \begin{bmatrix}0\\ \bot\end{bmatrix} \xrightarrow{a} \begin{bmatrix}0\\ 0\end{bmatrix} \xrightarrow{b} \begin{bmatrix}0\\ \bot\end{bmatrix} \cdots$$

- We can determine if $\begin{bmatrix}n_1\\ n_2\end{bmatrix} \xrightarrow{l} \begin{bmatrix}n_1'\\ n_2'\end{bmatrix}$ may appear in a ranking by just looking at n_1, n_2, n_1', n_2' and l : ranks should not increase.

First draft for $\bar{A}$

- For a two-state A (more states analogous):
  - States: all $\begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$ where accepting states get even rank
  - Initial states: all states of the form $\begin{bmatrix} n_1 \\ \bot \end{bmatrix}$
  - Transitions: all $\begin{bmatrix} n_1 \\ n_2 \end{bmatrix} \xrightarrow{a} \begin{bmatrix} n_1' \\ n_2' \end{bmatrix}$ s.t . ranks don´t increase
- The runs of the automaton on a word w correspond to all the rankings of $dag(w)$.
- Observe: $\bar{A}$ is a NBA even if A is a DBA, because there are many rankings for the same word.

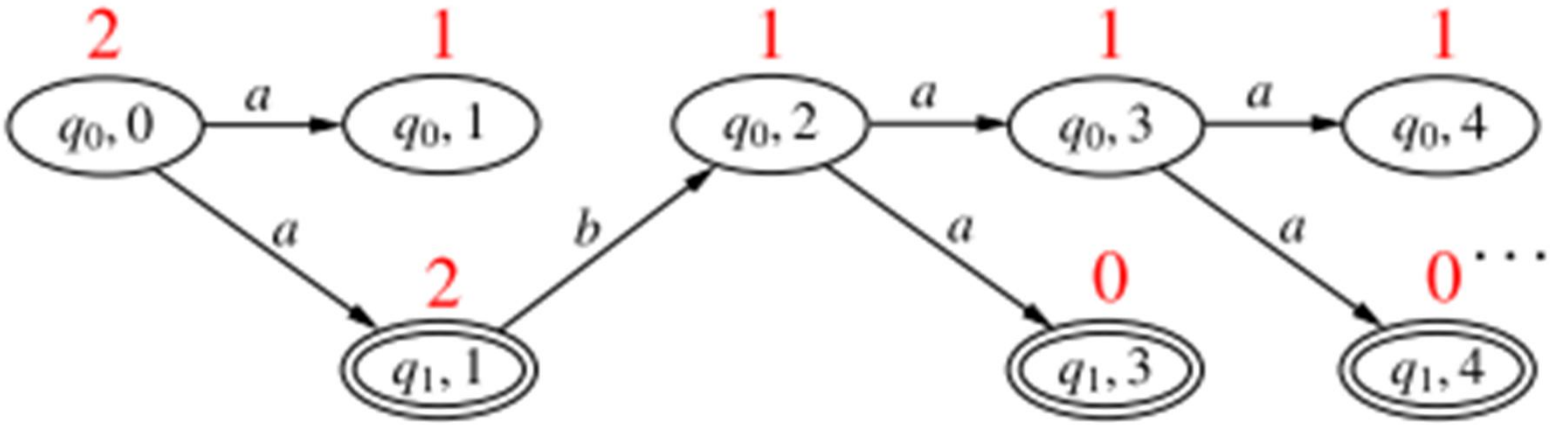
Problems to solve

- How to choose the accepting states?
  - They should be chosen so that a run is accepted iff its corresponding ranking is odd.
- Potentially infinitely many states (because rankings can contain arbitrarily large numbers)

Solving the first problem

- We use owing states and breakpoints again:
  - A breakpoint of a ranking is now a level of the ranking such that no state of the level owes a visit to a node of odd rank.
  - We have again: a ranking is odd iff it has infinitely many breakpoints.
  - We enrich the state with a set of owing states, and choose the accepting states as those in which the set is empty.

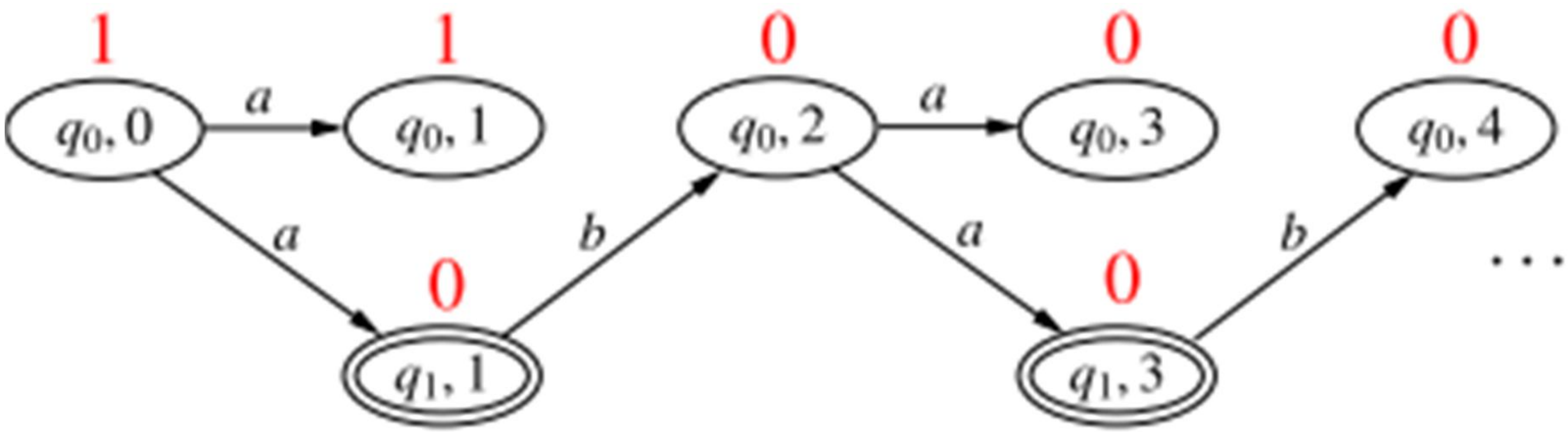
Owing states



$$\begin{bmatrix} 2 \\ \bot \end{bmatrix} \xrightarrow{a} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \xrightarrow{b} \begin{bmatrix} 1 \\ \bot \end{bmatrix} \xrightarrow{a} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \xrightarrow{a} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdots$$

$\{q_0\}$ $\{q_1\}$ $\emptyset$ $\{q_1\}$ $\emptyset$

Owing rankings



$$\begin{bmatrix}1\\ \bot\end{bmatrix} \xrightarrow{a} \begin{bmatrix}1\\ 0\end{bmatrix} \xrightarrow{b} \begin{bmatrix}0\\ \bot\end{bmatrix} \xrightarrow{a} \begin{bmatrix}0\\ 0\end{bmatrix} \xrightarrow{b} \begin{bmatrix}0\\ \bot\end{bmatrix} \cdots$$

$\emptyset$ $\{q_1\}$ $\{q_0\}$ $\{q_0, q_1\}$ $\{q_0\}$

Second draft for $\bar{A}$

- For a two-state A (the case of more states is analogous):
  - States: all pairs $\begin{bmatrix} n_1 \\ n_2 \end{bmatrix}, O$ wher accepting states get even rank, and O is set of owing states (of even rank)
  - Initial states: all $\begin{bmatrix} n_1 \\ \bot \end{bmatrix}, \{q_0\}$ where n_1 even if q_0 accepting.
  - Transitions: all $\begin{bmatrix} n_1 \\ n_2 \end{bmatrix}, O \xrightarrow{a} \begin{bmatrix} n_1' \\ n_2' \end{bmatrix}, O'$ s.t. ranks don't increase and owing states are correctly updated
  - Final states: all states $\begin{bmatrix} n_1 \\ n_2 \end{bmatrix}, \emptyset$

- The runs of $\bar{A}$ on a word w correspond to all the rankings of $dag(w)$.
- The accepting runs of $\bar{A}$ on a word w correspond to all the odd rankings of $dag(w)$.
- Therefore: $L(\bar{A}) = \overline{L(A)}$

Solving the second problem

Proposition: If w is rejected by A, then $dag(w)$ has an odd ranking in which ranks are taken from the range $[0,2n]$, where n is the number of states of A. Further, the initial node gets rank $2n$.

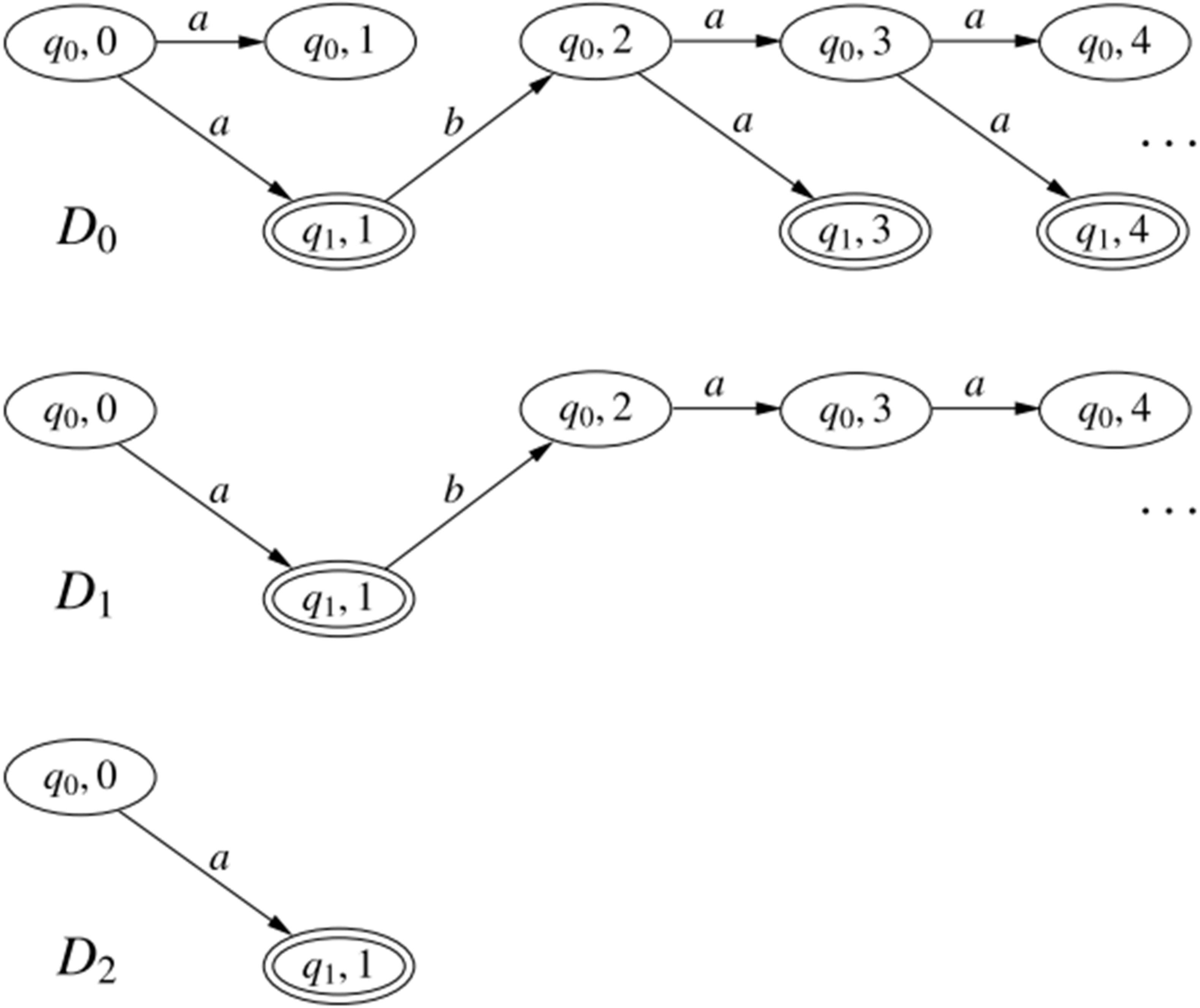
Proof: We construct such a ranking as follows:

- we proceed in $n + 1$ rounds (from round 0 to round n), each round with two steps $k.0$ and $k.1$ with the exception of round n which only has $n.0$
- each step removes a set of nodes together with all its descendants.
- the nodes removed at step $i.j$ get rank $2i + j$
- the rank of the initial node is increased to $2n$ if necessary (preserves the properties of rankings).

The steps

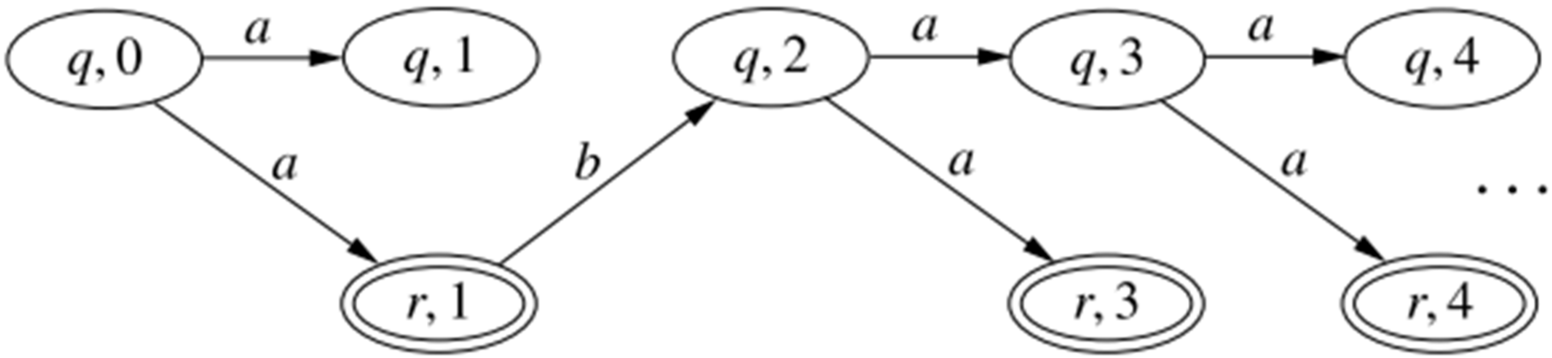
- Step $i.0$: remove all nodes having only finitely many successors.
- Step $i.1$: remove nodes that are non-accepting and have no accepting descendants

- This immediately guarantees :
  1. Ranks along a path cannot increase.
  2. Accepting states get even ranks, because they can only be removed at step $i.0$
- It remains to prove: no nodes left after $n + 1$ rounds .



q0, 0
a
q0, 1
a
q1, 1
b
q0, 2
a
q0, 3
a
q0, 4
a
q1, 3
a
q1, 4
...
D0
q0, 0
a
q1, 1
b
q0, 2
a
q0, 3
a
q0, 4
...
D1
q0, 0
a
q1, 1
D2

- To prove: no nodes left after n rounds .
- Each level of a dag has a width



- We define the width of a dag as the largest level width that appears infinitely often.
- Each round decreases the width of the dag by at least 1.
- Since the intial width is at most n after at most n rounds the width is 0, and then step $n.0$ removes all nodes.

Final $\bar{A}$

- For a two-state A (the case of more states is analogous):
  - States: all pairs $\begin{bmatrix} n_1 \\ n_2 \end{bmatrix}, O$ where O set of owing states and accepting states get even rank
  - Initial state: all $\begin{bmatrix} 2n \\ \perp \end{bmatrix}, \{q_0\}$
  - Transitions: all $\begin{bmatrix} n_1 \\ n_2 \end{bmatrix}, O \xrightarrow{a} \begin{bmatrix} n_1' \\ n_2' \end{bmatrix}, O'$ s.t. ranks don't increase and owing states are correctly updated
  - Final states: all states $\begin{bmatrix} n_1 \\ n_2 \end{bmatrix}, \emptyset$

An example

- We construct the complements of
  $A_1 = (\{q\}, \{a\}, \delta, \{q\}, \{q\})$ with $\delta(q, a) = \{q\}$
  $A_2 = (\{q\}, \{a\}, \delta, \{q\}, \emptyset)$ with $\delta(q, a) = \{q\}$
- States of A_1:
  $\langle 0, \emptyset \rangle, \langle 2, \emptyset \rangle, \langle 0, \{q\} \rangle, \langle 2, \{q\} \rangle$
- States of A_2:
  $\langle 0, \emptyset \rangle, \langle 1, \emptyset \rangle, \langle 2, \emptyset \rangle, \langle 0, \{q\} \rangle, \langle 2, \{q\} \rangle$
- Initial state of A_1 and A_2: $\langle 2, \{q\} \rangle$

An example

- Transitions of A_1:
  $\langle 2, \{q\}\rangle \xrightarrow{a} \langle 2, \{q\}\rangle$, $\langle 2, \{q\}\rangle \xrightarrow{a} \langle 0, \emptyset\rangle$, $\langle 0, \{q\}\rangle \xrightarrow{a} \langle 0, \{q\}\rangle$
- Transitions of A_2:
  $\langle 2, \{q\}\rangle \xrightarrow{a} \langle 2, \{q\}\rangle$, $\langle 2, \{q\}\rangle \xrightarrow{a} \langle 1, \emptyset\rangle$, $\langle 2, \{q\}\rangle \xrightarrow{a} \langle 0, \emptyset\rangle$,
  $\langle 1, \emptyset\rangle \xrightarrow{a} \langle 1, \emptyset\rangle$, $\langle 1, \emptyset\rangle \xrightarrow{a} \langle 0, \{q\}\rangle$,
  $\langle 0, \{q\}\rangle \xrightarrow{a} \langle 0, \{q\}\rangle$
- Final states of A_1: $\langle 0, \emptyset\rangle$, $\langle 2, \emptyset\rangle$ (unreachable)
- Final states of A_2: $\langle 0, \emptyset\rangle$, $\langle 1, \emptyset\rangle$, $\langle 2, \emptyset\rangle$ (only $\langle 1, \emptyset\rangle$ is reachable)

CompNBA(A)

Input: NBA $A = (Q, \Sigma, \delta, q_0, F)$

Output: NBA $\overline{A} = (\overline{Q}, \Sigma, \overline{\delta}, \overline{q}_0, \overline{F})$ with $L_\omega(\overline{A}) = \overline{L_\omega(A)}$

$\overline{Q}, \overline{\delta}, \overline{F} \leftarrow \emptyset$

$\overline{q}_0 \leftarrow [lr_0, \{q_0\}]$

$W \leftarrow \{\ [lr_0, \{q_0\}]\ \}$

while $W \neq \emptyset$ **do**

pick $[lr, P]$ **from** W; **add** $[lr, P]$ **to** $\overline{Q}$

if $P = \emptyset$ **then add** $[lr, P]$ **to** $\overline{F}$

for all $a \in \Sigma, lr' \in \mathfrak{R}$ such that $lr \overset{a}{\mapsto} lr'$ **do**

if $P \neq \emptyset$ **then** $P' \leftarrow \{q \in \delta(P, a) \mid lr'(q) \text{ is even }\}$

else $P' \leftarrow \{q \in Q \mid lr'(q) \text{ is even }\}$

add $([lr, P], a, [lr', P'])$ **to** $\overline{\delta}$

if $[lr', P'] \notin \overline{Q}$ **then add** $[lr', P']$ **to** W

return $(\overline{Q}, \Sigma, \overline{\delta}, \overline{q}_0, \overline{F})$

Complexity

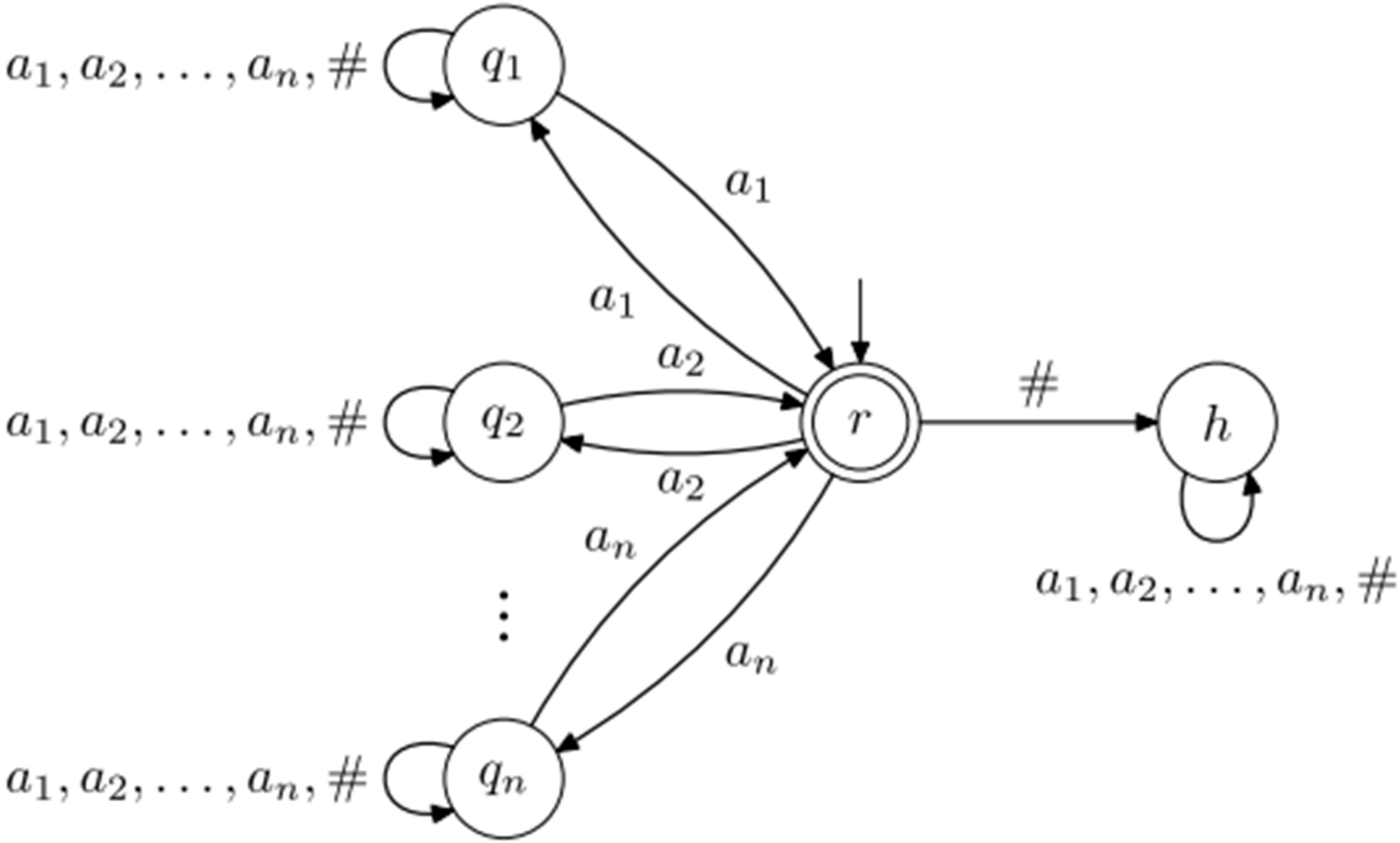
- A state consists of a level of a ranking and a set of owing states.
- A level assigns to each state a number f $[0,2n]$ or the symbol $\bot$.
- So the complement NBA has at most $(2n+2)^n \cdot 2^n \in n^{O(n)} = 2^{O(n \log n)}$ states.
- Compare with 2^n for the NFA case.
- We show that the $\log n$ factor is unavoidable.

We define a family $\{L_n\}_{n\geq 1}$ of ω-languages s.t.

- L_n is accepted by a NBA with $n+2$ states.
- Every NBA accepting $\overline{L_n}$ has at least $n! \in 2^{\Theta(n \log n)}$ states.

- The alphabet of L_n is $\Sigma_n = \{1,2, \ldots, n, \#\}$.
- Assign to a word $w \in \Sigma_n$ a graph $G(w)$ as follows:
  - Vertices: the numbers $1,2, \ldots, n$.
  - Edges: there is an edge $i \rightarrow j$ iff w contains infinitely many occurrences of ij.
- Define: $w \in L_n$ iff $G(w)$ has a cycle.

- L_n is accepted by a NBA with $n + 2$ states.



Every NBA accepting $\overline{L_n}$ has at least $n! \in 2^{\Theta(n \log n)}$ states.

- Let τ denote a permutation of $1, 2, \ldots, n$.
- We have:
  a) For every τ, the word $(\tau\ \#)^{\omega}$ belongs to $\overline{L_n}$ (i.e., its graph contains no cycle).
  b) For every two distinct τ_1, τ_2, every word containing inf. many occurrences of τ_1 and inf. many occurrences of τ_2 belongs to L_n.

Every NBA accepting $\overline{L_n}$ has at least $n! \in 2^{\Theta(n \log n)}$ states.

- Assume A recognizes $\overline{L_n}$ and let τ_1, τ_2 distinct. By (a), A has runs ρ_1, ρ_2 accepting $(\tau\_1\ \#)^\omega$, $(\tau_2\ \#)^\omega$. The sets of accepting states visited i.o. by ρ_1, ρ_2 are disjoint.
  - Otherwise we can ``interleave‘‘ ρ_1, ρ_2 to yield an acepting run for a word with inf. Many occurrences of τ_1, τ_2 , contradicting (b).
- So A has at least one accepting state for each permutation, and so at least $n!$ States.