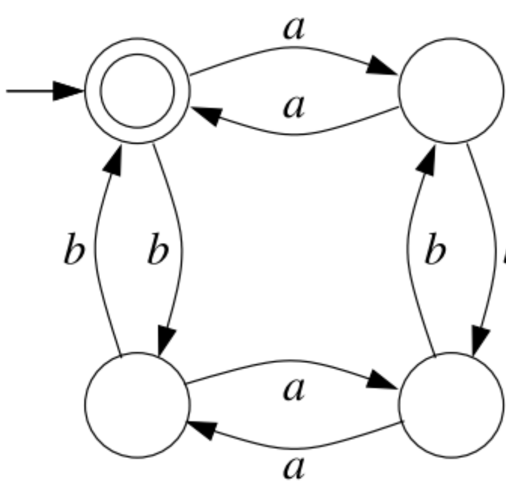
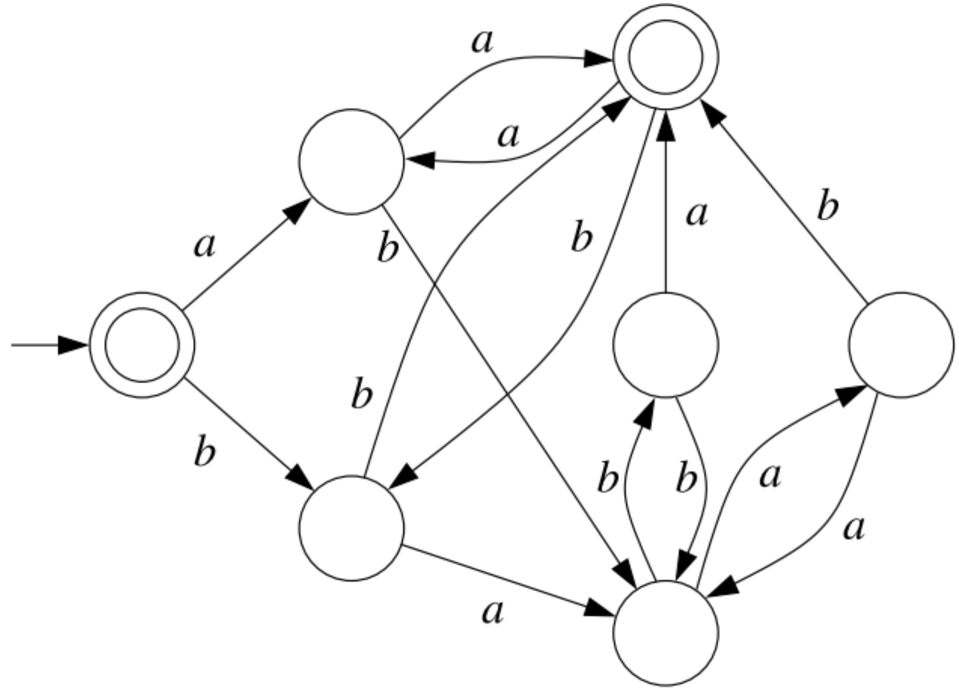
Minimization and Reduction





Residuals

- The residual of a language $L \subseteq \Sigma^*$ with respect to a word $w \in \Sigma^*$ is the language

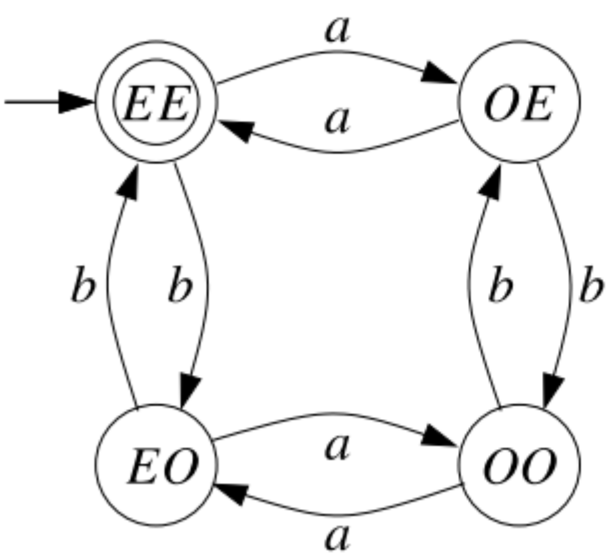
$$L^w = \{u \in \Sigma^* \mid wu \in L\}$$

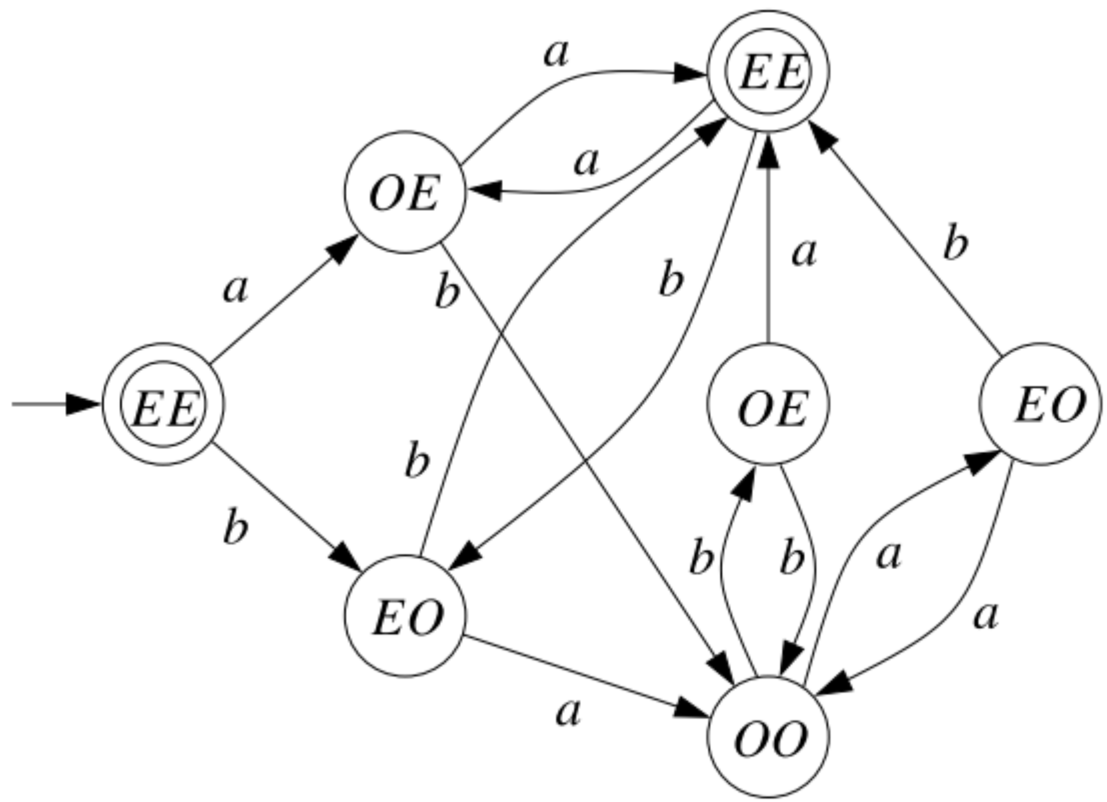
- A language $L' \subseteq \Sigma^*$ is a residual of L if $L' = L^w$ for at least one word $w \in \Sigma^*$
- Observe:
  - $L^\epsilon = L$
  - $(L^w)^v = L^{wv}$

Relation between residuals and states

- Let A be a (finite or infinite) deterministic automaton.
- **Def:** The language of a state q of A, denoted by $L_A(q)$ or just $L(q)$, is the language recognized by A with q as initial state.
- **Observation 1**: State-languages are residuals.
  - For every state q of A: $L(q)$ is a residual of $L(A)$.
- **Observation 2**: Residuals are state-languages.
  - For every residual R of $L(A)$: there is a state q such that $R = L(q)$.

Relation between residuals and states





Relation between residuals and states

- Important consequence:

  Regular languages have finitely many residuals.

  Languages with infinitely many residuals are not regular.

Canonical DA for a language

- Let $L \subseteq \Sigma^*$ be a language (not necessarily regular). The canonical DA for L is the tuple

$$C_L = (Q_L, \Sigma, \delta_L, q_{0L}, F_L)$$

  where

  – Q_L is the set of residuals of L, i.e., $Q_L = \{ L^w \mid w \in \Sigma^* \}$

  – $\delta(R, a) = R^a$ for every residual $R \in Q_L$ and $a \in \Sigma$

  – $q_{0L} = L$

  – $F_L = \{R \in Q_L \mid \epsilon \in R\}$

Canonical DA for a language

- For the language $EE \subseteq \{a, b\}^*$:

$$Q_{EE} =$$

$$q_{0EE} =$$

$$F_{EE} =$$

$$\delta_{EE} =$$

Canonical DA for a language

- For the language $a^*b^* \subseteq \{a,b\}^*$:

$$Q_{a^*b^*} =$$

$$q_{0(a^*b^*)} =$$

$$F_{a^*b^*} =$$

$$\delta_{a^*b^*} =$$

Canonical DA for a language

- Proposition. C_L recognizes L.
- Proof. We prove by induction on $|w|$: $w \in L$ iff $w \in L(C_L)$

If $|w| = 0$ then $w = \varepsilon$, and we have

$$
\begin{array}{lll}
 & \varepsilon \in L & (w = \epsilon) \\
\Leftrightarrow & L \in F_L & (\text{definition of } F_L) \\
\Leftrightarrow & q_{0L} \in F_L & (q_{0L} = L) \\
\Leftrightarrow & \varepsilon \in L(C_L) & (q_{0L} \text{ is the initial state of } C_L)
\end{array}
$$

If $|w| > 0$, then $w = aw'$ for some $a \in \Sigma$ and $w' \in \Sigma^*$, and we have

$$
\begin{array}{lll}
 & aw' \in L & \\
\Leftrightarrow & w' \in L^a & (\text{definition of } L^a) \\
\Leftrightarrow & w' \in L(C_{L^a}) & (\text{induction hypothesis}) \\
\Leftrightarrow & aw' \in L(C_L) & (\delta_L(L, a) = L^a)
\end{array}
$$

Canonical DA for a language

Theorem. If L is regular, then C_L is the unique minimal DFA up to isomorphism recognizing L.

Proof.

1. C_L is a DFA for L with a minimal number of states.
   - C_L has as many states as residuals.
   - Every DFA for L has at least as many states as residuals
2. Every minimal DFA for L is isomorphic to C_L.

   Let A be an arbitrary minimal DFA for L. Then:
   - The states of A are in bijection with the residuals of L.
   - The transitions of A are completely determined by this bijection: if $q \leftrightarrow L^w$, then $\delta(q, a) \leftrightarrow L^{wa}$
   - The initial state is the state in bijection with L.
   - The final states are those in bijection with residuals containing ϵ.

Canonical DA for a language

Corollary. A DFA is minimal iff $L(q) \neq L(q')$ for every two distinct states q and q'.

Proof.

($\Rightarrow$): Let A be a minimal DFA.

Every residual of $L(A)$ is recognized by at least one state of A (holds for every DFA).

Since A is minimal, it has as many states as C_L, and so its number of states is equal to the number of residuals of $L(A)$.

Therefore: distinct states of A recognize distinct residuals of $L(A)$.

Canonical DA for a language

Corollary. A DFA is minimal iff $L(q) \neq L(q')$ for every two distinct states q and q'.

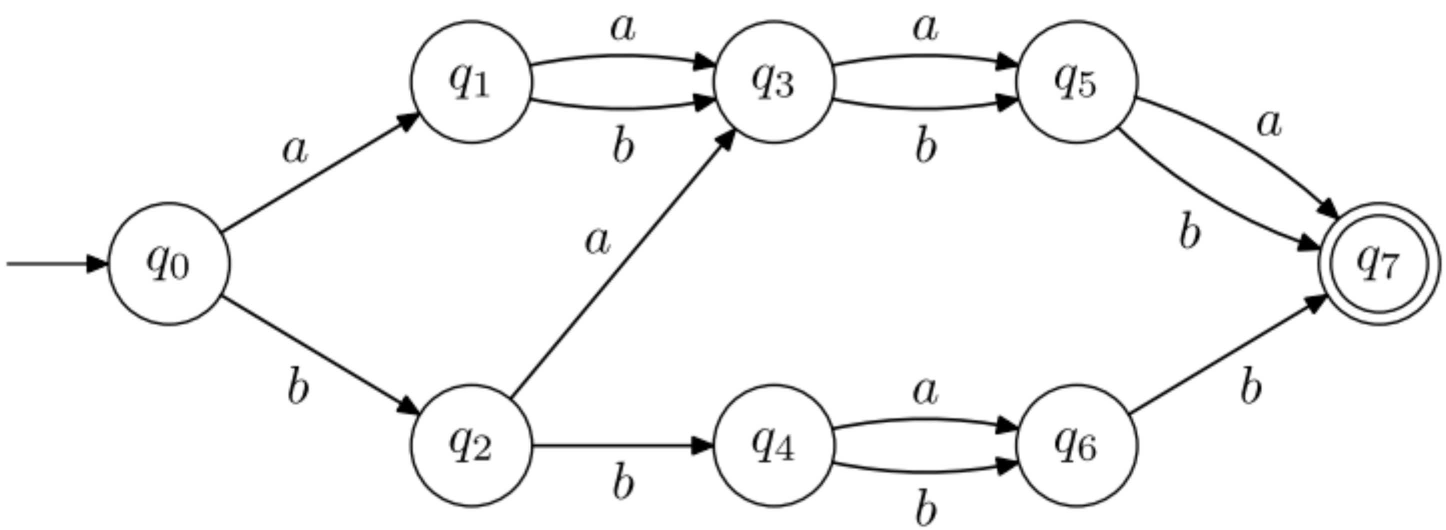
Proof.

(⇐): Let A be a DFA such that distinct states recognize distinct languages.

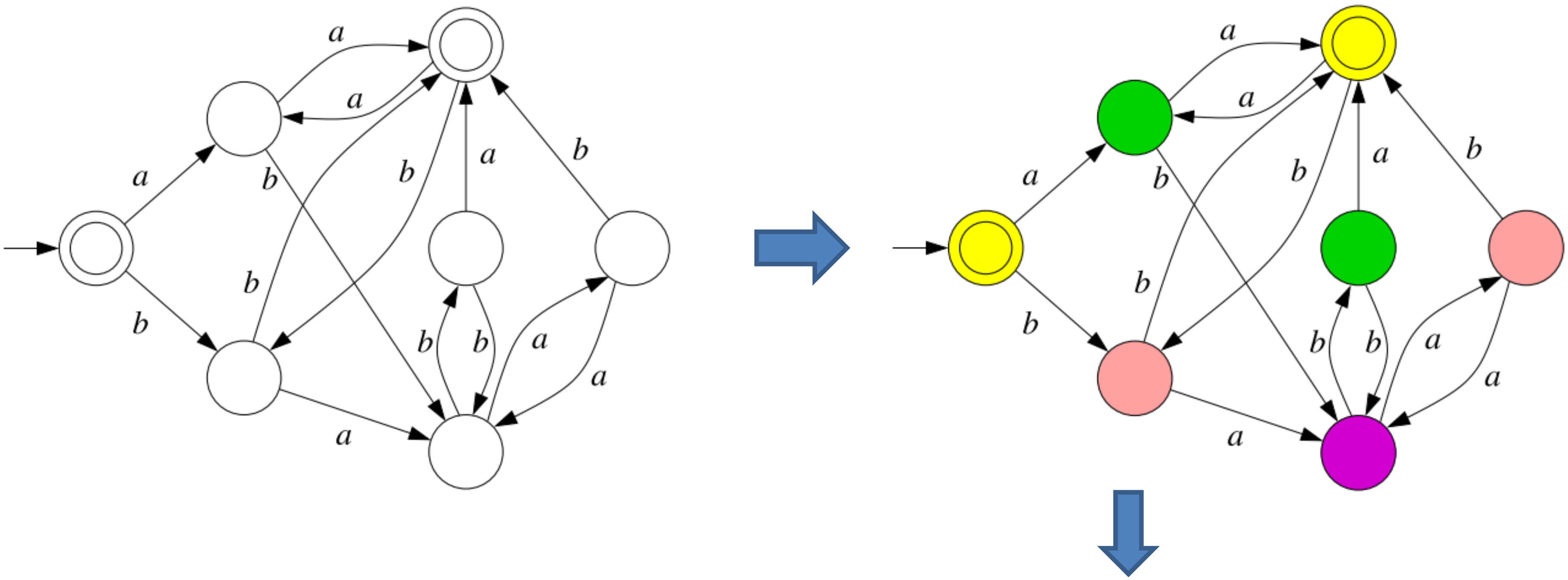
Since every state of A recognizes a residual of $L(A)$, and every residual of $L(A)$ is recognized by some state of A (holds for every DFA), the number of states of A is equal to the number of residuals of $L(A)$.

So A has as many states as C_L, and so it is minimal.

Is it minimal ?

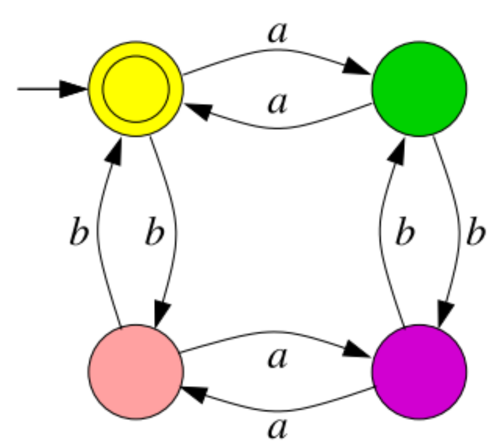


Minimizing DFAs



Plan for the next slides:

1. Computing the language partition
2. Quotienting
3. Thm: The result is the minimal DFA



Computing the language partition

State partitions

- Block: set of states.
- Partition: set of blocks such that each state belongs to exactly one block.
- Partition P refines partition P' if every block of P is contained in some block of P'.
- If P refines P', then we say that P is finer than P', and P' is coarser than P.
- Language partition: the partition in which two states belong to the same block iff they recognize the same language.

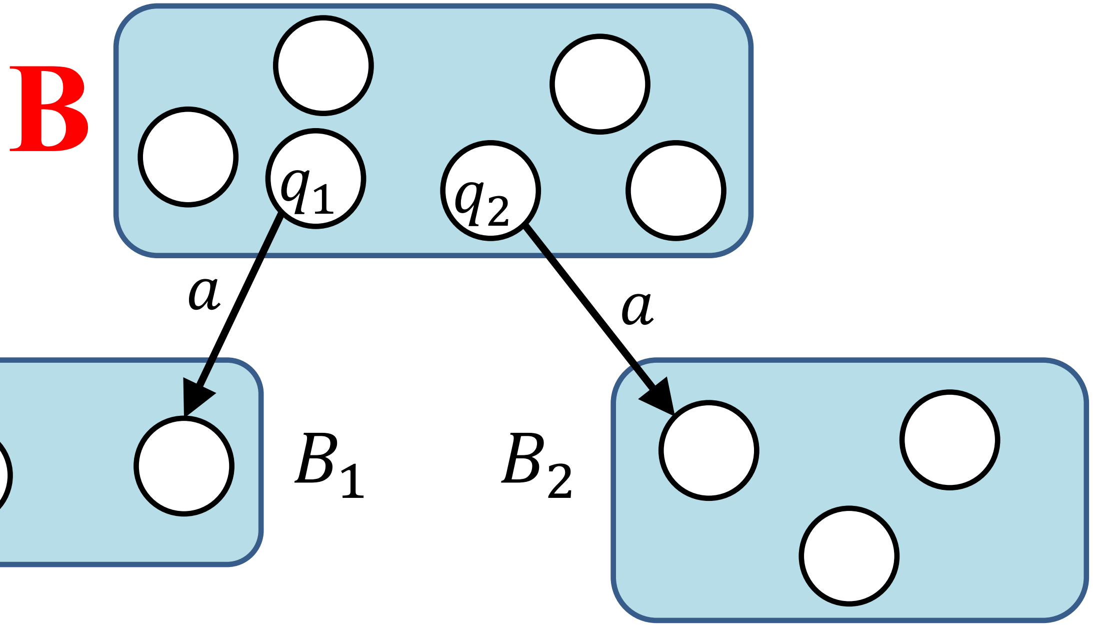
Computing the language partition

- Start with the partition containing (one or) two blocks:
  - Block 1: Final states (accept ε)
  - Block 2: Non-final states (do not accept ε)
- Iteratively split blocks, ensuring that states recognizing the same language always stay in the same block.
- Blocks that contain at least two states recognizing different languages are called **unstable**.

Computing the language partition

Finding an unstable block

If two states q_1, q_2 belong to the same block B
but $\delta(q_1, a)$ and $\delta(q_2, a)$ belong to different blocks for some $a \in \Sigma$,
then B is **unstable**.

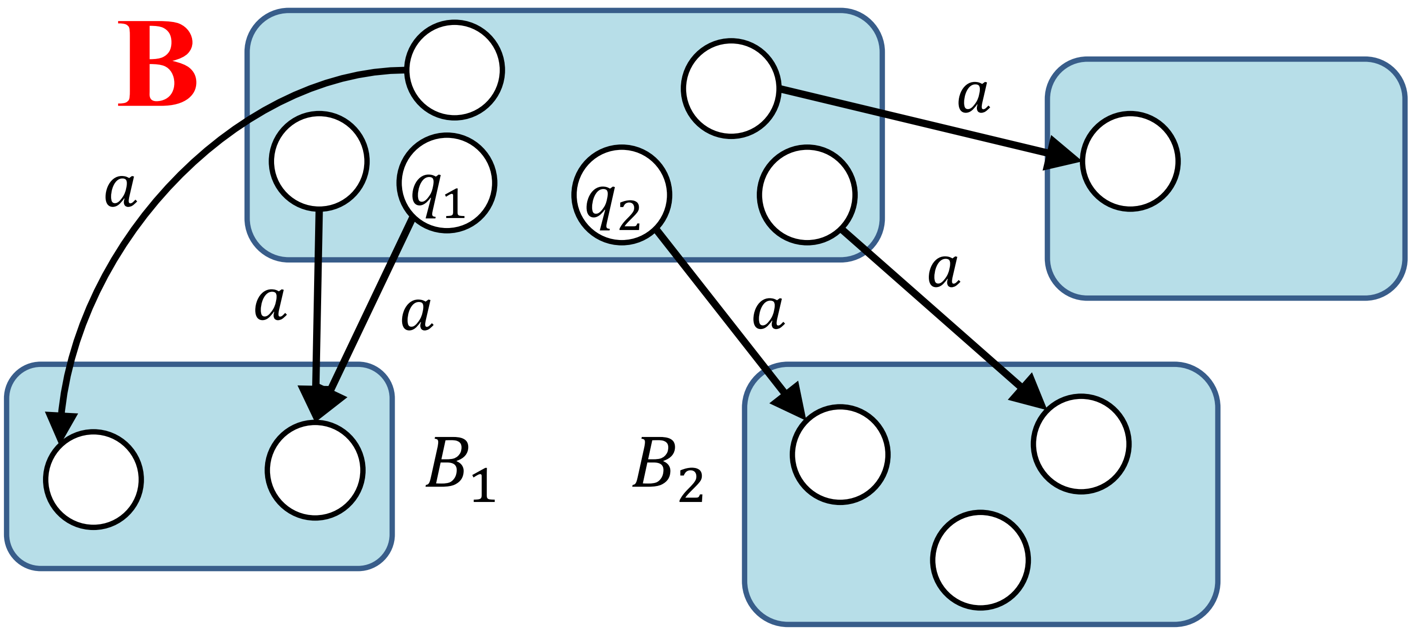


Computing the language partition

Splitting an **unstable** block

We say that (a, B_1) and (a, B_2) are splitters of B.

A splitter (a, B') splits B into two blocks: states q such that $\delta(q, a) \in B'$, and the rest.



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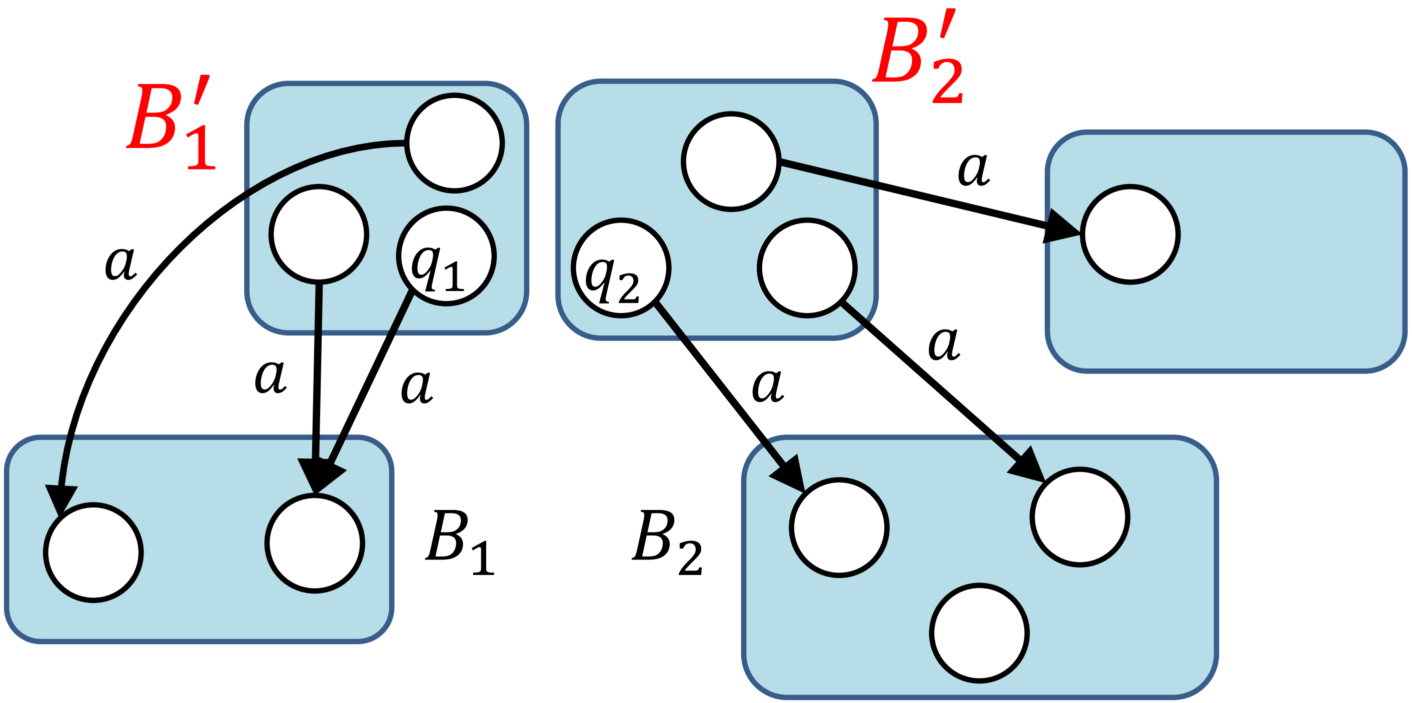


Correctness

- The algorithm terminates.

  Every execution of the loop increases the number of blocks by 1, and the number of blocks is bounded by the number of states.

- After termination, two states belong to the same block iff they recognize the same language.

  We show that after termination:

  (1) If two states belong to different blocks, they recognize different languages.

  (2) If two states recognize different languages, they belong to different blocks.

Correctness

(1) If two states q_1 and q_2 belong to different blocks, they recognize different languages.

By induction on the number k of splittings until q_1 and q_2 are split (put into different blocks).

- $k = 0$. Then q_1 is final and q_2 non-final, or vice versa, and we are done.
- $k \rightarrow k + 1$. Then there are q_1', q_2' such that $q_1 \xrightarrow{a} q_1'$, $q_2 \xrightarrow{a} q_2'$, and q_1' , q_2' have been split before q_1, q_2 are split.

  By induction hypothesis q_1' and q_2' recognize different languages.

  Since the automaton is a DFA, q_1 and q_2 also recognize different languages.

Correctness

(2) If two states q_1 and q_2 recognize different languages, they belong to different blocks.

Let w be a shortest word that belongs to, say, $L(q_1)$ but not to $L(q_2)$. By induction on the length of w.

- $|w| = 0$. Then $w = \varepsilon$, q_1 is final, and q_2 is non-final. So q_1 and q_2 belong to different blocks from the start.
- $|w| > 0$. Then $w = aw'$ for some a, w'. Let $q_1' = \delta(q_1, a)$ and $q_2' = \delta(q_2, a)$. Then $L(q_1') \neq L(q_2')$ by the DFA property.

  By induction hypothesis q_1', q_2' are put at some some point into different blocks.

  If at this point q_1 and q_2 still belong to the same block, then the block becomes unstable and is eventually split.

Quotienting

Quotient w.r.t. a partition

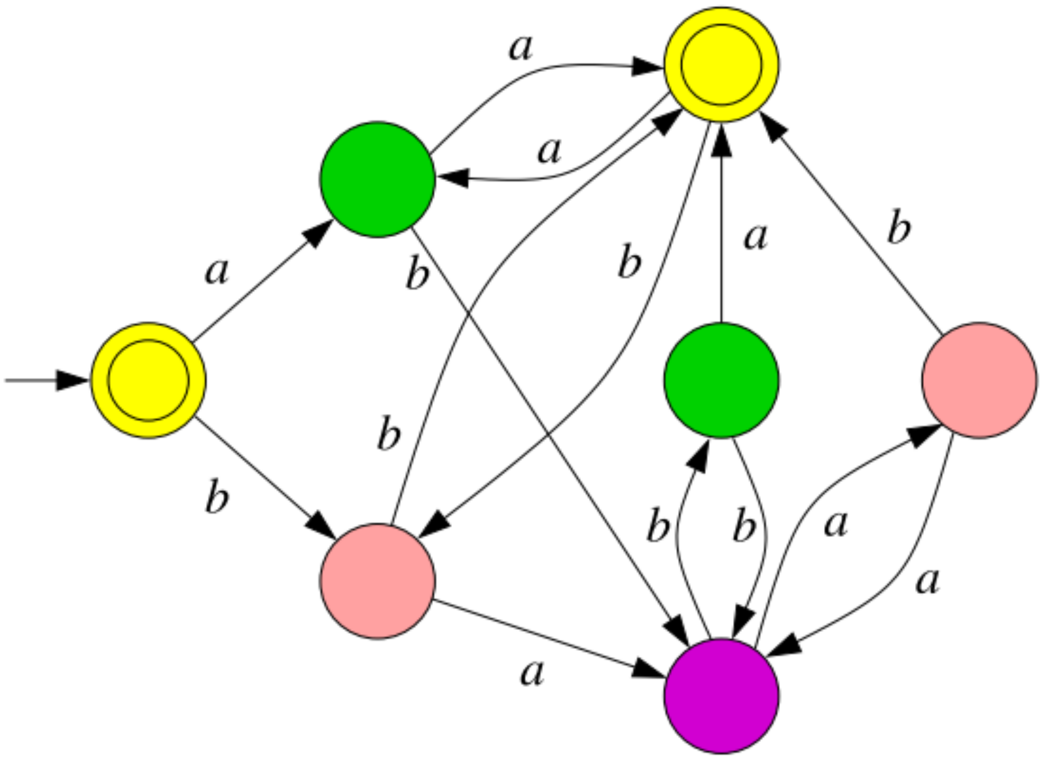
- Definition: The quotient of a NFA $A = (Q, \Sigma, \delta, q_0, F)$ with respect to a partition P is the NFA

$$A/P = (Q_P, \Sigma, \delta_P, q_{0P}, F_P)$$

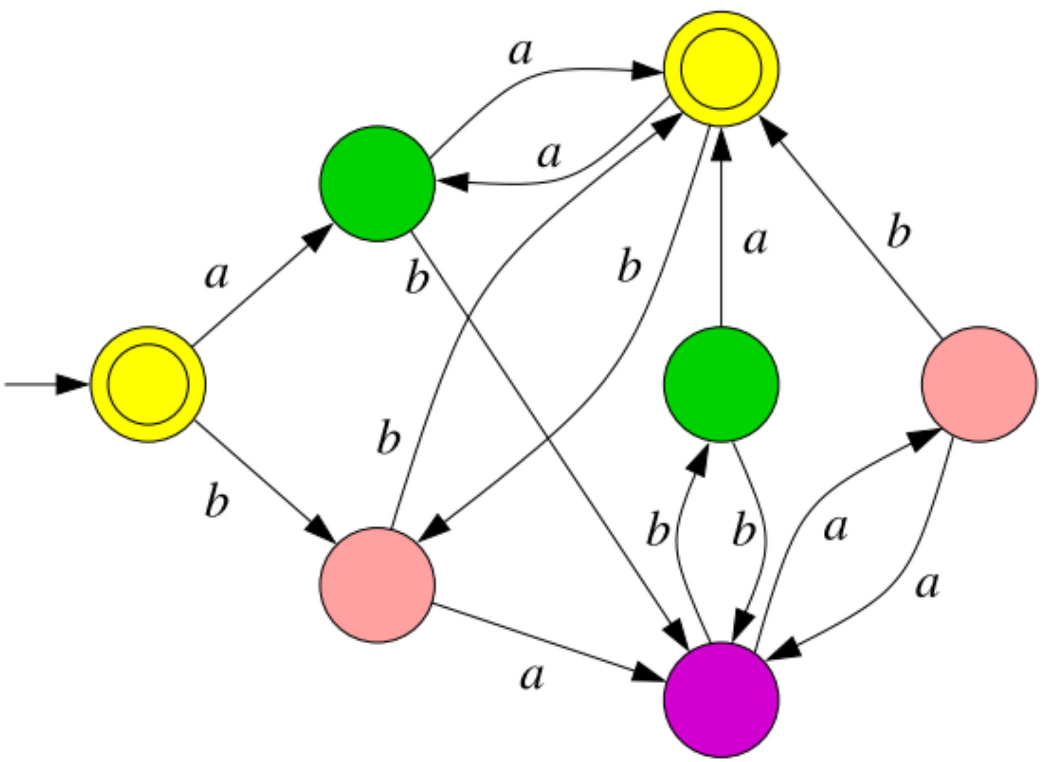
where

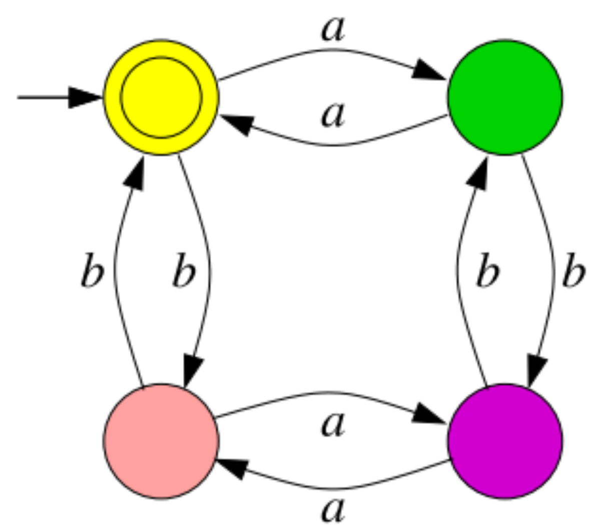
  - $Q_P = P$
  - $(B, a, B') \in \delta_P$ iff $(q, a, q') \in \delta$ for some $q \in B$ and some $q' \in B'$
  - q_{0P} is the block containing q_0
  - F_P is the set of blocks that contain some state of F

Quotient w.r.t. a partition



Quotient w.r.t. a partition





Quotient w.r.t. a partition

Proposition: The quotient of a DFA with respect to its language partition is (isomorphic to) the canonical DFA.

The proof has two parts:

(1) A DFA and its quotient w.r.t. the language partition recognize the same language.
(2) The quotient is minimal (and therefore the canonical DFA).

Quotient w.r.t. a partition

(1) A DFA and its quotient w.r.t. the language partition recognize the same language.

We prove a more general result (for later use):

Lemma: Let A be a NFA, and let P be any partition that refines the language partition P_l.

a) For every state q: $L_A(q) = L_{\mathrm{A/P}}(B)$, where B is the block containing q.
b) If A is a DFA and $P = P_l$, then A/P is also a DFA.

Quotient w.r.t. a partition

a) For every state q of A: $L_A(q) = L_{A/P}(B)$, where B is the block containing q.

We prove that for every word $w \in \Sigma$:

$$w \in L_A(q) \Leftrightarrow w \in L_{A/P}(B).$$

By induction on $|w|$.

- $|w| = 0$. Then $w = \varepsilon$ and

$$\begin{aligned} \epsilon \in L_A(q) \ &\text{iff} \ \ q \in F \\ &\text{iff} \ \ B \subseteq F \qquad \text{(because } P \text{ refines } P_\ell\text{)} \\ &\text{iff} \ \ B \in F_P \\ &\text{iff} \ \ \epsilon \in L_{A/P}(B) \end{aligned}$$

Quotient w.r.t. a partition

a) For every state q of A: $L_A(q) = L_{A/P}(B)$, where B is the block containing q.

- $|w| > 0$. Then $w = aw'$.

  There is $q \xrightarrow{a} q'$ in A such that $w' \in L_A(q')$.

  There is $B \xrightarrow{a} B'$ in A/P such that $q' \in B'$.

  We have:

$aw' \in L_A(q)$	iff	$w' \in L_A(q')$	(Def. of q)
	iff	$w' \in L_{A/P}(B')$	(induction hyp.)
	iff	$aw' \in L_{A/P}(B)$	($B \xrightarrow{a} B'$)

Quotient w.r.t. a partition

b) If A is a DFA and $P = P_l$, then A/P is also a DFA.

We show: If $B \xrightarrow{a} B_1$ and $B \xrightarrow{a} B_2$, then $B_1 = B_2$.

- There are $q, q' \in B$, $q_1 \in B_1$, $q_2 \in B_2$ such that $q \xrightarrow{a} q_1$ and $q' \xrightarrow{a} q_2$.
- Since $P \subseteq P_l$, q and q' recognize the same language.
- Since A is a DFA, q_1 and q_2 recognize the same language.
- Since $P \subseteq P_l$, $B_1 = B_2$.

Quotient w.r.t. a partition

2) The quotient of a DFA A w.r.t. the language partition is the canonical DFA.

- By 1.b, the quotient is a DFA.
- By 1.a, applied to the initial state, A/P_ℓ recognizes the same language as A.
- Since the quotient is w.r.t. the language partition, different blocks of the quotient recognize different languages. So A/P is minimal.

Hopcroft´s algorithm

- The algorithm for the computation of the language partition is nondeterministic: It does not specify which unstable block to split next.
- Hopcroft´s algorithm is a refinement that carefully chooses the split order, using the inverse of the transition function and achieves a complexity of $O(mn \log n)$ for a DFA with n states over an m-letter alphabet.
- The algorithm maintains a workset of possible splitters.

Hopcroft´s algorithm

- The algorithm maintains a workset of candidate splitters (a, B).
- When a candidate (a, B) is taken from the workset, it is applied to all current blocks.
- Observation 1: After applying (a, B) to all blocks it never brings anything to apply it again

  $\Rightarrow$ it is safe to ensure that candidates removed from the workset are never added to the workset again.
- Observation 2: If B is split into B_0 and B_1, then splitting w.r.t. any two of $(a, B), (a, B_0), (a, B_1)$ produces the same result as splitting with respect to all three.

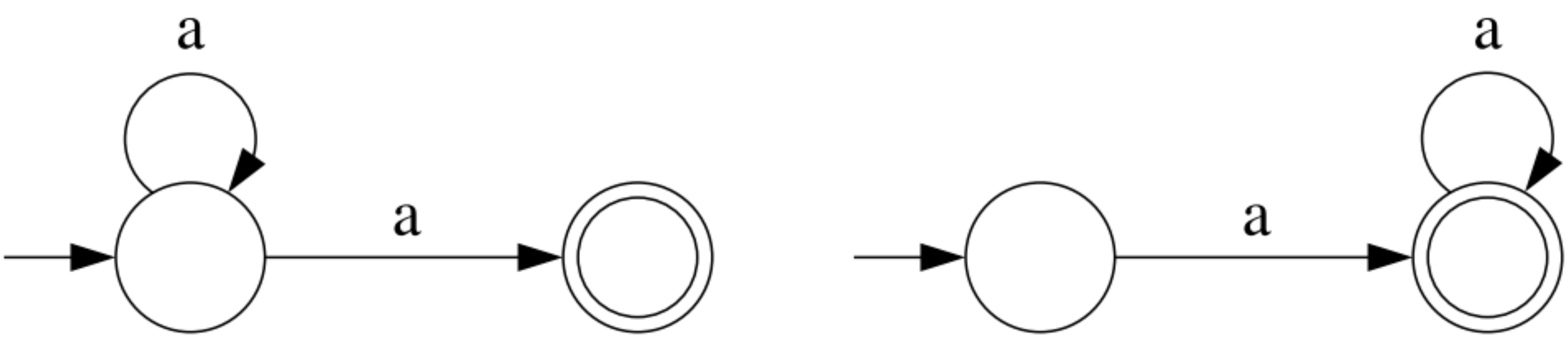
Hopcroft´s algorithm

Hopcroft(A)
Input: DFA $A = (Q, \Sigma, \delta, q_0, F)$
Output: The language partition P_ℓ.

if $F = \emptyset$ or $Q \setminus F = \emptyset$ **then return** $\{Q\}$
else $P \leftarrow \{F, Q \setminus F\}$
$\mathcal{W} \leftarrow \{\ (a, \min\{F, Q \setminus F\})\ \mid a \in \Sigma\}$
while $\mathcal{W} \neq \emptyset$ **do**
  pick (a, B') from $\mathcal{W}$
  for all $B \in P$ split by (a, B') **do**
    replace B by B_0 and B_1 in P
    for all $b \in \Sigma$ **do**
      if $(b, B) \in \mathcal{W}$ **then** replace (b, B) by (b, B_0) and (b, B_1) in $\mathcal{W}$
      else add $(b, \min\{B_0, B_1\})$ to $\mathcal{W}$
return P

Reducing NFAs

Minimal NFAs are not unique



Finding minimal NFAs is hard

Theorem: The following problem is PSPACE-complete: Given an NFA A and a number k, decide if there is another NFA B equivalent to A and having at most k states.

Proof idea: We will show later that the following problem is PSPACE complete: given an NFA A over alphabet Σ, decide whether $L(A) = \Sigma^*$.

The problem above can be reduced to this one. This shows PSPACE-hardness.

Reducing NFAs

We wish to use the same idea as before:

- Compute a suitable partition P of the states of the NFA.
- Quotient the NFA with respect to this partition.

Requirements on P :

- $L(A) = L(A/P)$
- Efficiently computable

Partitions suitable for reduction

- Recall: For every NFA A and partition P that refines the language partition: $L(A) = L(A/P)$.
- So any such partition is good for reduction.
- A partition refines the language partition iff states in the same block recognize the same language (states in different blocks may not recognize different languages, though!).
- (Observe: Such partitions refine the partition $\{F, Q \setminus F\}$.)

Computing a suitable partition

- Idea: use the same algorithm as for DFA, but with new notions of **unstable** block and block splitting.
- We must guarantee:

  after termination, states of a block recognize the same language

  or, equivalently

  after termination, states recognizing different languages belong to different blocks

The key observation

If $L(q_1) \neq L(q_2)$ then either

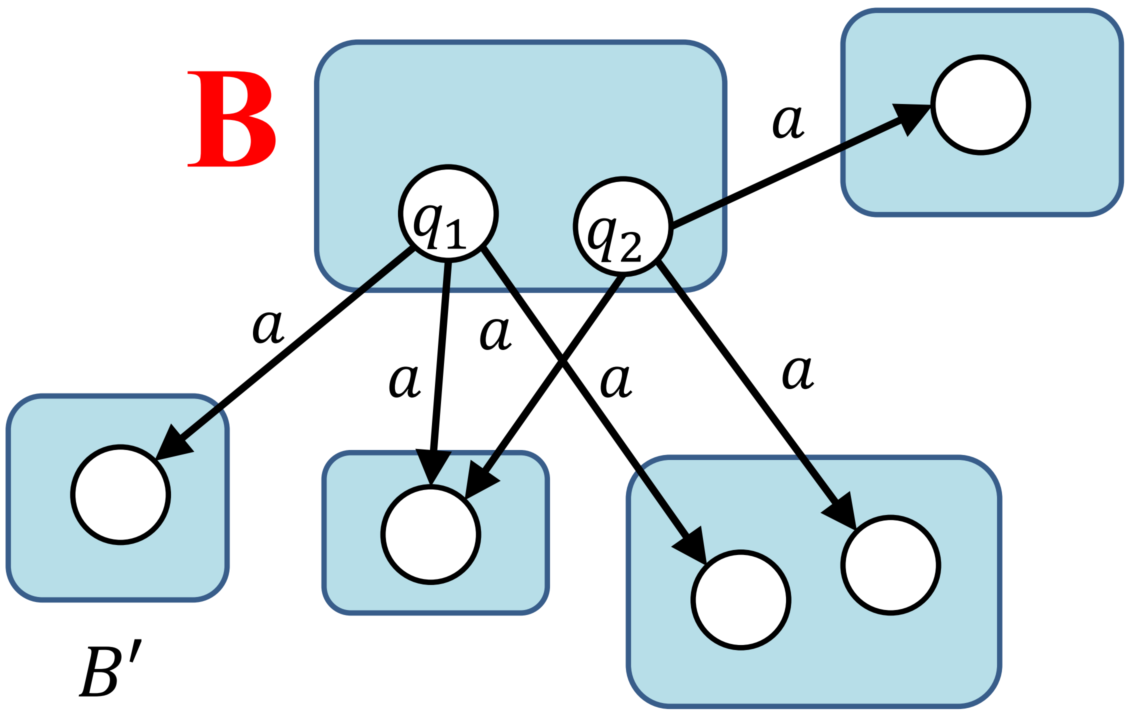
- one of q_1, q_2 is final and the other non-final, or
- one of q_1, q_2, say q_1, has a transition $q_1 \xrightarrow{a} q_1'$ such that every a-transition $q_2 \xrightarrow{a} q_2'$ satisfies: $L(q_1') \neq L(q_2')$.

Unstable blocks

A block B is **unstable** if there are states $q_1, q_2 \in B$, a block B' and $a \in \Sigma$ such that

$$\delta(q_1, a) \cap B' \neq \emptyset \quad \text{and} \quad \delta(q_2, a) \cap B' = \emptyset$$

We say that (a, B') splits B.

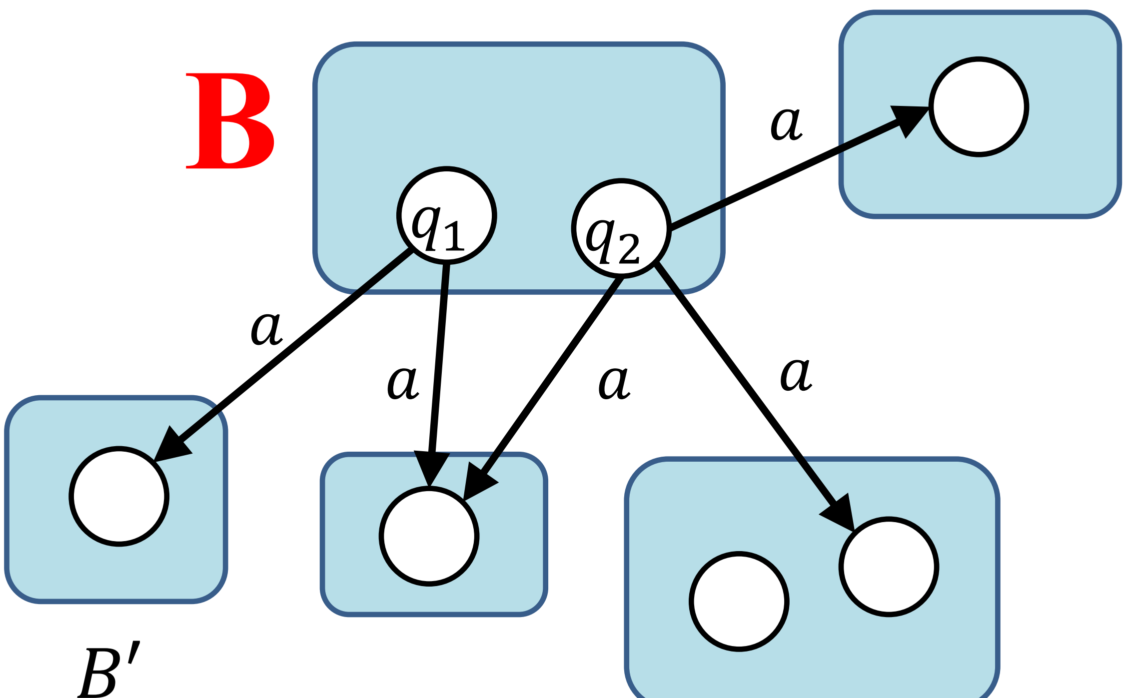


Splitting blocks

Splitting an unstable block

We say that (a, B') is a splitter of B.

A splitter (a, B') splits B into two blocks: states q such that $\delta(q, a) \cap B' \neq \emptyset$, and the rest.



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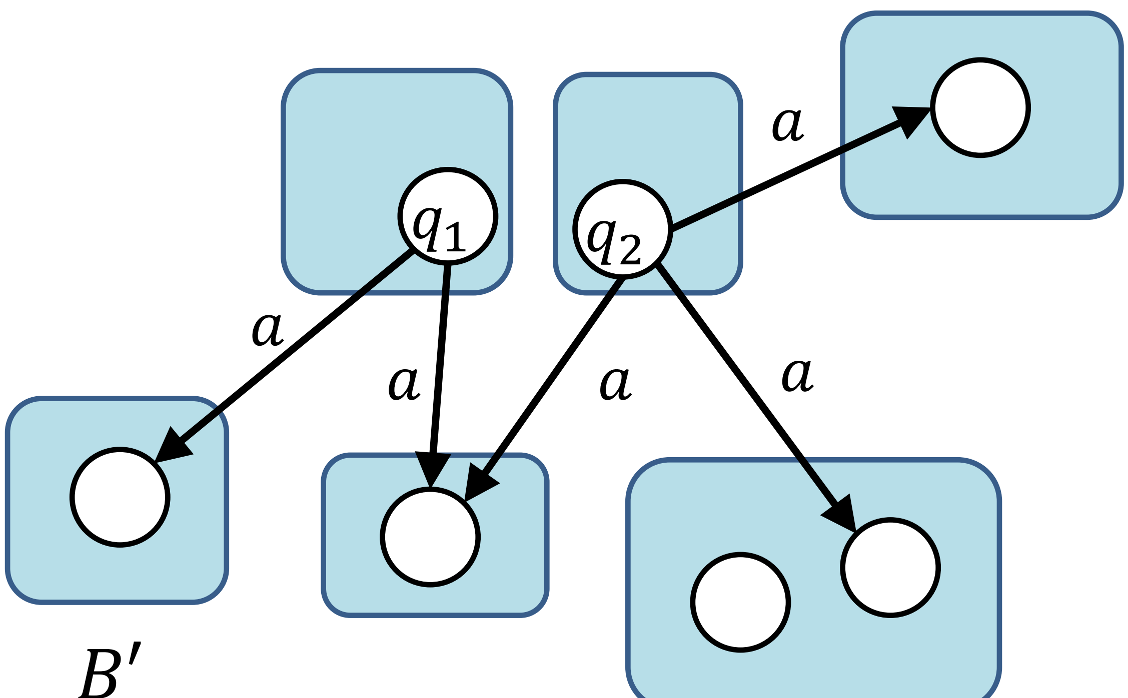
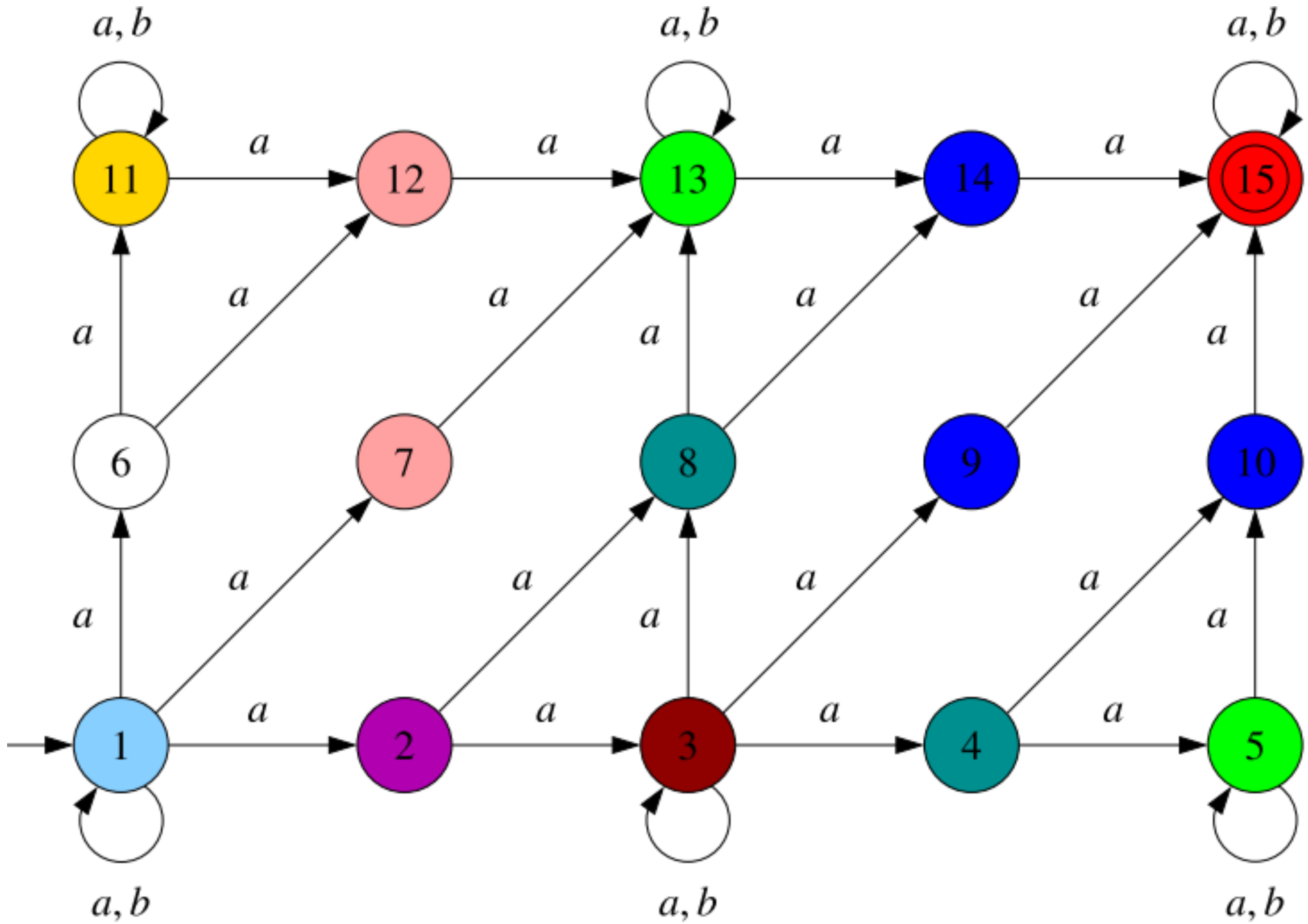


An example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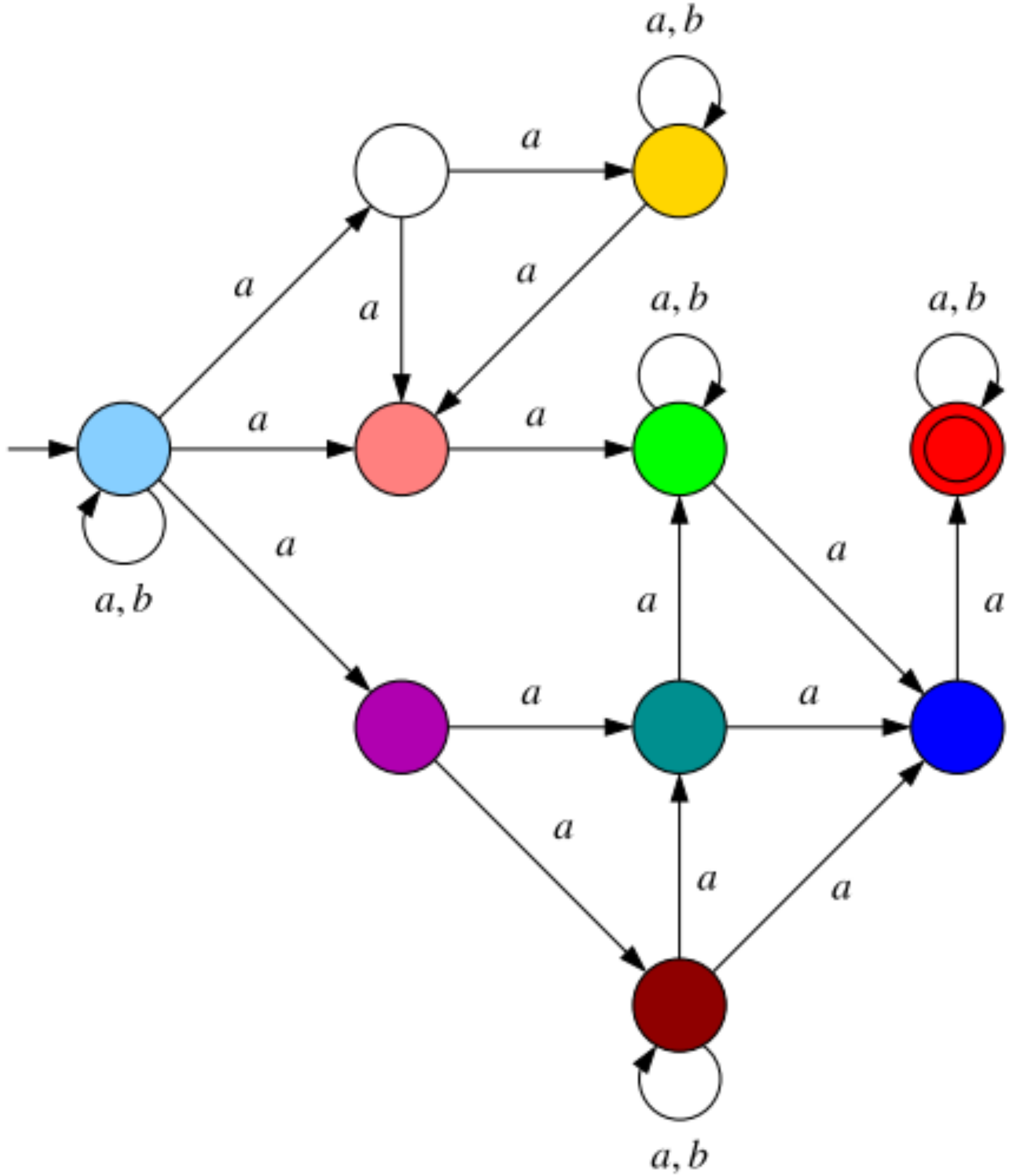
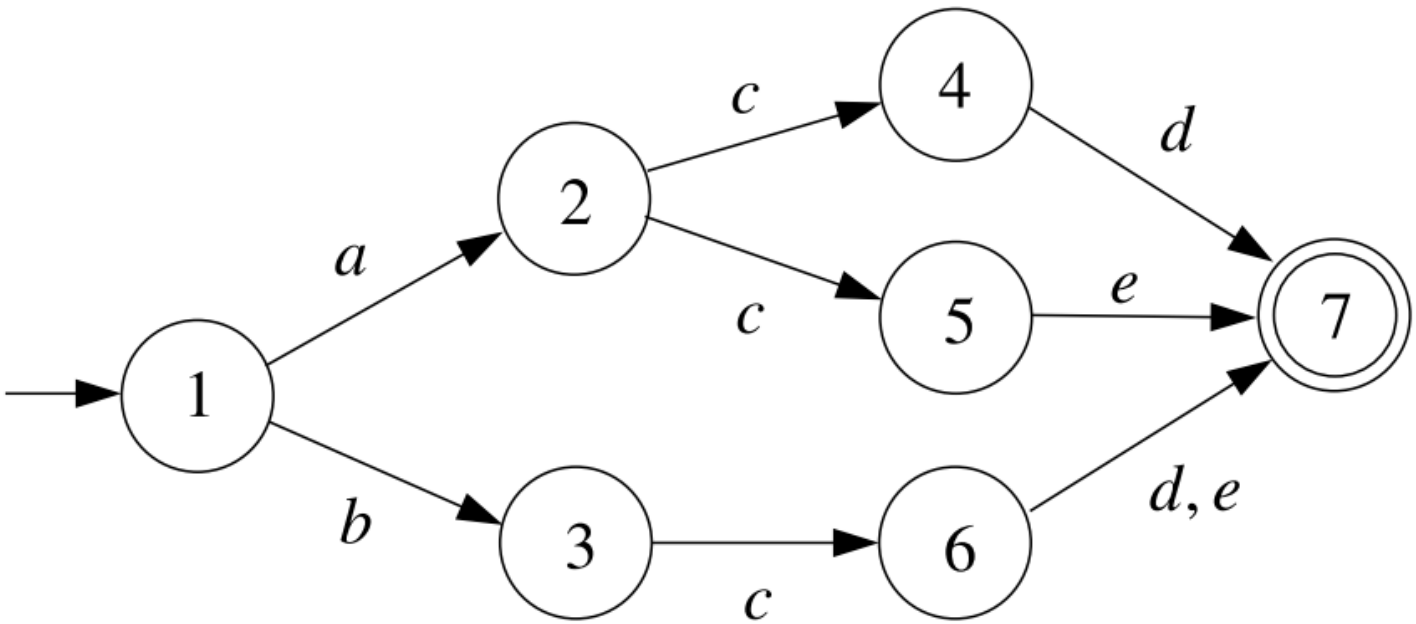
An example



The algorithm not always computes the language partition



States 2 and 3 recognize the same language: $c(d + e)$
However, the algorithm puts them into different blocks.