Automata and Formal Languages — Exercise Sheet 16

Exercise 16.1
Prove or disprove:

(a) $\mathbf{GF}(\varphi \vee \psi) \equiv \mathbf{GF}\varphi \vee \mathbf{GF}\psi$

(b) $\mathbf{GF}(\varphi \wedge \psi) \equiv \mathbf{GF}\varphi \wedge \mathbf{GF}\psi$

(c) $(\varphi \vee \psi)\ \mathbf{U}\ \rho \equiv (\varphi\ \mathbf{U}\ \rho) \vee (\psi\ \mathbf{U}\ \rho)$

(d) $\rho\ \mathbf{U}\ (\varphi \vee \psi) \equiv (\rho\ \mathbf{U}\ \varphi) \vee (\rho\ \mathbf{U}\ \psi)$

Exercise 16.2
Let $A = (Q, \Sigma, \delta, q_0, F)$ be an automaton such that $Q = P \times [n]$ for some finite set P and $n \geq 1$. Automaton A models a system made of n processes. A state $(p, i) \in Q$ represents the current global state p of the system, and the last process i that was executed.

We define two predicates exec_j and enab_j over Q indicating whether process j is respectively executed and enabled. More formally, for every $q = (p, i) \in Q$ and $j \in [n]$, let

$$\begin{aligned} \text{exec}_j(q) &\iff i = j, \\ \text{enab}_j(q) &\iff (p, i) \to (p', j) \text{ for some } p' \in P. \end{aligned}$$

(a) Give LTL formulas over Q^ω for the following statements:

   (i) All processes are executed infinitely often.
   (ii) If a process is enabled infinitely often, then it is executed infinitely often.
   (iii) If a process is eventually permanently enabled, then it is executed infinitely often.

(b) The three above properties are known respectively as *unconditional*, *strong* and *weak* fairness. Show the following implications, and show that the reverse implications do not hold:

$$\text{unconditional fairness} \implies \text{strong fairness} \implies \text{weak fairness}.$$

Exercise 16.3

Let $\text{AP} = \{p, q\}$ and let $\Sigma = 2^{\text{AP}}$. An LTL formula is a tautology if it is satisfied by all computations. Which of the following LTL formulas are tautologies?

(a) $\mathbf{G}p \rightarrow \mathbf{F}p$

(b) $\mathbf{G}(p \rightarrow q) \rightarrow (\mathbf{G}p \rightarrow \mathbf{G}q)$

(c) $\mathbf{FG}p \vee \mathbf{FG}\neg p$

(d) $\neg\mathbf{F}p \rightarrow \mathbf{F}\neg\mathbf{F}p$

(e) $(\mathbf{G}p \rightarrow \mathbf{F}q) \leftrightarrow (p \;\mathbf{U}\; (\neg p \vee q))$

(f) $\neg(p \;\mathbf{U}\; q) \leftrightarrow (\neg p \;\mathbf{U}\; \neg q)$

(g) $\mathbf{G}(p \rightarrow \mathbf{X}p) \rightarrow (p \rightarrow \mathbf{G}p)$

Solution 16.1

(a) True. If $\sigma \models \mathbf{GF}\varphi \vee \mathbf{GF}\psi$, then $\sigma \models \mathbf{GF}(\varphi \vee \psi)$. If $\sigma \models \mathbf{GF}(\varphi \vee \psi)$, then there exist $i_0 < i_1 < \cdots$ such that

$$\sigma^{i_j} \models \varphi \vee \psi \text{ for every } j \in \mathbb{N}. \tag{1}$$

Let $I = \{j \in \mathbb{N} : \sigma^{i_j} \models \varphi\}$ and $J = \{j \in \mathbb{N} : \sigma^{i_j} \models \psi\}$. If I and J are both finite, then (1) does not hold, which is a contradiction. Therefore, at least one of I and J is infinite. This implies that $\sigma \models \mathbf{GF}\varphi \vee \mathbf{GF}\psi$. $\square$

(b) False. Let $\sigma = (\{p\}\{q\})^\omega$. We have $\sigma \not\models \mathbf{GF}(p \wedge q)$ and $\sigma \models \mathbf{GF}p \wedge \mathbf{GF}q$.

(c) False. Let $\sigma = \{p\}\{q\}\{r\}\emptyset^\omega$. We have $\sigma \models (p \vee q)\ \mathbf{U}\ r$ and $\sigma \not\models (p\ \mathbf{U}\ r) \vee (q\ \mathbf{U}\ r)$.

(d) True, since:

$$\begin{aligned}
\sigma \models \rho\ \mathbf{U}\ (\varphi \vee \psi) &\iff \exists k \geq 0 \text{ s.t. } \sigma^k \models (\varphi \vee \psi) \wedge \forall 0 \leq i < k\ \sigma^i \models \rho \\
&\iff \exists k \geq 0 \text{ s.t. } ((\sigma^k \models \varphi) \vee (\sigma^k \models \psi)) \wedge \forall 0 \leq i < k\ \sigma^i \models \rho \\
&\iff \exists k \geq 0 \text{ s.t. } (\sigma^k \models \varphi \wedge \forall 0 \leq i < k\ \sigma^i \models \rho) \vee (\sigma^k \models \psi \wedge \forall 0 \leq i < k\ \sigma^i \models \rho) \\
&\iff (\exists k \geq 0 \text{ s.t. } \sigma^k \models \varphi \wedge \forall 0 \leq i < k\ \sigma^i \models \rho) \vee (\exists k \geq 0 \text{ s.t. } \sigma^k \models \psi \wedge \forall 0 \leq i < k\ \sigma^i \models \rho) \\
&\iff \sigma \models (\rho\ \mathbf{U}\ \varphi) \vee (\rho\ \mathbf{U}\ \psi).
\end{aligned}$$

$\square$

Solution 16.2

(a) (i) $\bigwedge_{j \in [n]} \mathbf{GF}\ \text{exec}_j$

(ii) $\bigwedge_{j \in [n]} (\mathbf{GF}\ \text{enab}_j \rightarrow \mathbf{GF}\ \text{exec}_j)$

(iii) $\bigwedge_{j \in [n]} (\mathbf{FG}\ \text{enab}_j \rightarrow \mathbf{GF}\ \text{exec}_j)$

(b)

- Unconditional fairness implies strong fairness. For the sake of contradiction, suppose unconditional fairness holds for some execution σ, but not strong fairness. By assumption, there exists $j \in [n]$ such that $\sigma \not\models (\mathbf{GF}\ \text{enab}_j \rightarrow \mathbf{GF}\ \text{exec}_j)$. Thus,

$$\begin{aligned}
\sigma &\not\models (\mathbf{GF}\ \text{enab}_j \rightarrow \mathbf{GF}\ \text{exec}_j) &&\iff \\
\sigma &\models \neg(\mathbf{GF}\ \text{enab}_j \rightarrow \mathbf{GF}\ \text{exec}_j) &&\iff \\
\sigma &\models \neg(\neg\mathbf{GF}\ \text{enab}_j \vee \mathbf{GF}\ \text{exec}_j) &&\iff \\
\sigma &\models \mathbf{GF}\ \text{enab}_j \wedge \neg\mathbf{GF}\ \text{exec}_j &&\implies \\
\sigma &\models \neg\mathbf{GF}\ \text{exec}_j
\end{aligned}$$

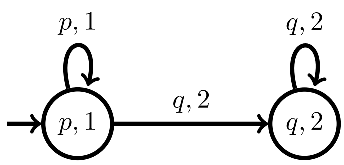
which contradicts unconditional fairness. $\square$

- Strong fairness implies weak fairness. For the sake of contradiction, suppose strong fairness holds for some execution σ, but not weak fairness. By assumption, there exists $j \in [n]$ such that $\sigma \not\models (\mathbf{FG}\ \text{enab}_j \rightarrow \mathbf{GF}\ \text{exec}_j)$. Thus,

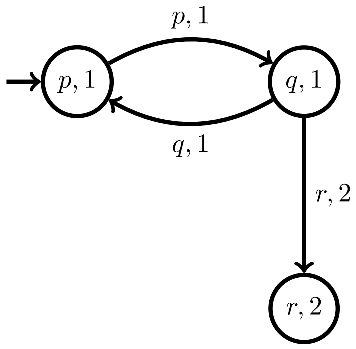
$$\begin{aligned}
\sigma &\not\models (\mathbf{FG}\ \text{enab}_j \rightarrow \mathbf{GF}\ \text{exec}_j) &&\iff \\
\sigma &\models \neg(\mathbf{FG}\ \text{enab}_j \rightarrow \mathbf{GF}\ \text{exec}_j) &&\iff \\
\sigma &\models \neg(\neg\mathbf{FG}\ \text{enab}_j \vee \mathbf{GF}\ \text{exec}_j) &&\iff \\
\sigma &\models \mathbf{FG}\ \text{enab}_j \wedge \neg\mathbf{GF}\ \text{exec}_j &&\implies \\
\sigma &\models \mathbf{GF}\ \text{enab}_j \wedge \neg\mathbf{GF}\ \text{exec}_j &&\iff \\
\sigma &\models \neg(\mathbf{GF}\ \text{enab}_j \rightarrow \mathbf{GF}\ \text{exec}_j) &&\iff \\
\sigma &\not\models \mathbf{GF}\ \text{enab}_j \rightarrow \mathbf{GF}\ \text{exec}_j
\end{aligned}$$

which contradicts strong fairness. $\square$

- Strong fairness does not imply unconditional fairness. Execution $(p,1)(q,2)^\omega$ of the automaton below satisfies strong fairness, but not unconditional fairness.



- Weak fairness does not imply strong fairness. Execution $((p,1)(q,1))^\omega$ of the automaton below satisfies weak fairness, but not strong fairness.



Solution 16.3

(a) $\mathbf{G}p \rightarrow \mathbf{F}p$ is a tautology since

$$\begin{aligned}\sigma \models \mathbf{G}p &\iff \forall k \geq 0\ \sigma^k \models p \\ &\implies \exists k \geq 0\ \sigma^k \models p \\ &\iff \sigma \models \mathbf{F}p.\end{aligned}$$

(b) $\mathbf{G}(p \rightarrow q) \rightarrow (\mathbf{G}p \rightarrow \mathbf{G}q)$ is a tautology. For the sake of contradiction, suppose this is not the case. There exists σ such that

$$\sigma \models \mathbf{G}(p \rightarrow q), \text{ and} \tag{2}$$
$$\sigma \not\models (\mathbf{G}p \rightarrow \mathbf{G}q). \tag{3}$$

By (3), we have

$$\begin{aligned}&\sigma \models \mathbf{G}p, \text{ and} \\ &\sigma \not\models \mathbf{G}q.\end{aligned}$$

Therefore, there exists $k \geq 0$ such that $p \in \sigma(k)$ and $q \notin \sigma(k)$ which contradicts (2).

(c) $\mathbf{FG}p \vee \mathbf{FG}\neg p$ is not a tautology since it is not satisfied by $(\{p\}\{q\})^\omega$.

(d) $\neg\mathbf{F}p \rightarrow \mathbf{F}\neg\mathbf{F}p$ is a tautology since $\varphi \rightarrow \mathbf{F}\varphi$ is a tautology for every formula φ.

(e) $(\mathbf{G}p \rightarrow \mathbf{F}q) \leftrightarrow (p\ \mathbf{U}\ (\neg p \vee q))$ is a tautology. We have

$$\begin{aligned}\mathbf{G}p \rightarrow \mathbf{F}q &\equiv \neg\mathbf{G}p \vee \mathbf{F}q && \text{(by def. of implication)} \\ &\equiv \mathbf{F}\neg p \vee \mathbf{F}q \\ &\equiv \mathbf{F}(\neg p \vee q) \\ &\equiv \mathbf{F}(p \rightarrow q) && \text{(by def. of implication)}\end{aligned}$$

Therefore, we have to show that

$$\mathbf{F}(p \rightarrow q) \leftrightarrow (p\ \mathbf{U}\ (p \rightarrow q)).$$

$\leftarrow$) Let σ be such that $\sigma \models (p\ \mathbf{U}\ (p \rightarrow q))$. In particular, there exists $k \geq 0$ such that $\sigma^k \models (p \rightarrow q)$. Therefore, $\sigma \models \mathbf{F}(p \rightarrow q)$.

$\rightarrow$) Let σ be such that $\sigma \models \mathbf{F}(p \rightarrow q)$. Let $k \geq 0$ be the smallest position such that $\sigma^k \models (p \rightarrow q)$. For every $0 \leq i < k$, we have $\sigma^i \not\models (p \rightarrow q)$ which is equivalent to $\sigma^i \models p \wedge \neg q$. Therefore, for every $0 \leq i < k$, we have $\sigma^i \models p$. This implies that $\sigma \models p\ \mathbf{U}\ (p \rightarrow q)$.

(f) $\neg(p\ \mathbf{U}\ q) \leftrightarrow (\neg p\ \mathbf{U}\ \neg q)$ is not a tautology. Let $\sigma = \{p\}\{q\}^\omega$. We have $\sigma \not\models \neg(p\ \mathbf{U}\ q)$ and $\sigma \models (\neg p\ \mathbf{U}\ \neg q)$.

(g) $\mathbf{G}(p \rightarrow \mathbf{X}p) \rightarrow (p \rightarrow \mathbf{G}p)$ is a tautology since

$$\begin{aligned}
\mathbf{G}(p \rightarrow \mathbf{X}p) \rightarrow (p \rightarrow \mathbf{G}p) &\equiv \neg\mathbf{G}(\neg p \vee \mathbf{X}p) \vee (\neg p \vee \mathbf{G}p) && \text{(by def. of implication)} \\
&\equiv \mathbf{F}(p \wedge \neg\mathbf{X}p) \vee \neg p \vee \mathbf{G}p \\
&\equiv \neg\mathbf{G}p \rightarrow (\neg p \vee (\mathbf{F}(p \wedge \mathbf{X}\neg p)) && \text{(by def. of implication)} \\
&\equiv \mathbf{F}\neg p \rightarrow (\neg p \vee (\mathbf{F}(p \wedge \mathbf{X}\neg p)) \\
&\equiv \mathbf{F}\neg p \rightarrow \mathbf{F}\neg p.
\end{aligned}$$