# Automata and Formal Languages — Exercise Sheet 12

**Exercise 12.1**
Prove or disprove:

(a) For every Büchi automaton $A$, there exists a Büchi automaton $B$ with a single initial state and such that $L_\omega(A) = L_\omega(B)$;

(b) For every Büchi automaton $A$, there exists a Büchi automaton $B$ with a single accepting state and such that $L_\omega(A) = L_\omega(B)$;

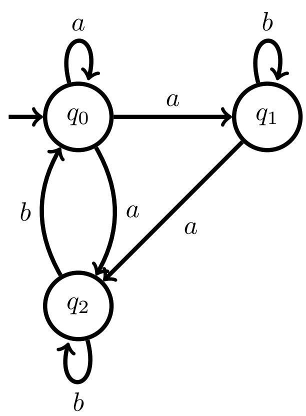
(c) There exists a Büchi automaton recognizing the finite $\omega$-language $\{w\}$ such that $w \in \{0, 1, \ldots, 9\}^\omega$ and $w_i$ is the $i^{\text{th}}$ decimal of $\sqrt{2}$.

**Exercise 12.2**
Give *deterministic* Rabin automata and Muller automata for the following language:

$$L = \{w \in \{a, b\}^\omega : w \text{ contains finitely many } a\text{'s}\}.$$

**Exercise 12.3**
Give a procedure that translates non-deterministic Rabin automata to non-deterministic Büchi automata.

**Exercise 12.4**
Consider the following automaton $A$:




(a) Interpret $A$ as a Muller automaton with acceptance condition $\{\{q_1\}, \{q_0, q_2\}\}$. Use algorithms *NMAtoNGA* and *NGAtoNBA* from the lecture notes to construct a Büchi automaton that recognizes the same language as $A$.

(b) Interpret $A$ as a Rabin automaton with acceptance condition $\{\langle\{q_0, q_2\}, \{q_1\}\rangle\}$. Follow the approach presented in class to construct a Büchi automaton that recognizes the same language as $A$.

**Solution 12.1**

(a) True. The construction for NFAs still work for Büchi automata.

Let $B = (Q, \Sigma, \delta, Q_0, F)$ be a Büchi automaton. We add a state to $Q$ which acts as the single initial state. More formally, we define $B' = (Q \cup \{q_{\text{init}}\}, \Sigma, \delta', \{q_{\text{init}}\}, F)$ where

$$\delta'(q, a) = \begin{cases} \bigcup_{q_0 \in Q_0} \delta(q_0, a) & \text{if } q = q_{\text{init}}, \\ \delta(q, a) & \text{otherwise.} \end{cases}$$

We have $L_\omega(B) = L_\omega(B')$, since there exists $q_0 \in Q_0$ such that

$$q_0 \xrightarrow{a_1}_B q_1 \xrightarrow{a_2}_B q_2 \xrightarrow{a_3}_B \cdots$$

if and only if

$$q_{\text{init}} \xrightarrow{a_1}_{B'} q_1 \xrightarrow{a_2}_{B'} q_2 \xrightarrow{a_3}_{B'} \cdots$$

(b) False. Let $L = \{a^\omega, b^\omega\}$. Suppose there exists a Büchi automaton $B = (Q, \{a, b\}, \delta, Q_0, F)$ such that $L_\omega(B) = L$ and $F = \{q\}$. Since $a^\omega \in L$, there exist $q_0 \in Q_0$, $m \geq 0$ and $n > 0$ such that

$$q_0 \xrightarrow{a^m} q \xrightarrow{a^n} q.$$

Similarly, since $b^\omega \in L$, there exist $q_0' \in Q_0$, $m' \geq 0$ and $n' > 0$ such that

$$q_0' \xrightarrow{b^{m'}} q \xrightarrow{b^{n'}} q.$$

This implies that

$$q_0 \xrightarrow{a^m} q \xrightarrow{b^{n'}} q \xrightarrow{b^{n'}} \cdots$$

Therefore, $a^m (b^{n'})^\omega \in L$, which is a contradiction. □

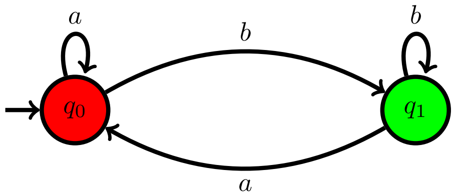
(c) False. Suppose there exists a Büchi automaton $B = (Q, \{0, 1, \ldots, 9\}, \delta, Q_0, F)$ such that $L_\omega(B) = \{w\}$. There exist $u \in \{0, 1, \ldots, 9\}^*$, $v \in \{0, 1, \ldots, 9\}^+$, $q_0 \in Q_0$ and $q \in F$ such that

$$q_0 \xrightarrow{u} q \xrightarrow{v} q.$$

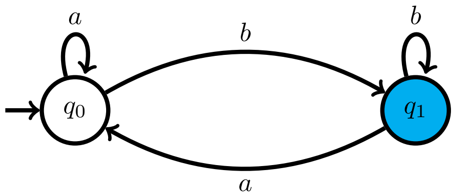
Therefore, $uv^\omega \in L_\omega(B)$ which implies that $w = uv^\omega$. Since $w$ represents the decimals of $\pi$, we conclude that $\pi$ is rational, which is a contradiction. □

**Solution 12.2**

- We give the following Rabin automaton with acceptance condition $\{(\{q_1\}, \{q_0\})\}$, i.e. where $q_1$ must be visited infinitely often and $q_0$ must be visited finitely often:




- We give the following Muller automaton with acceptance condition $\{\{q_1\}\}$, i.e. where precisely $\{q_1\}$ must be visited infinitely often:



**Solution 12.3**

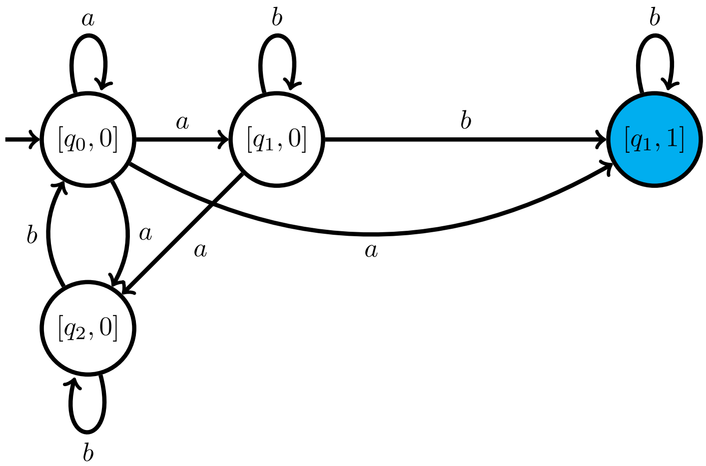
NBA can be easily transformed into nondterministic Rabin automata (NRA) and vice versa, without any exponential blow-up.

**NBA → NRA.** Just observe that a Büchi condition $\{q_1, \ldots, q_k\}$ is equivalent to the following Rabin condition $\{\, (\{q_1\}, \emptyset), \ldots, (\{q_n\}, \emptyset) \,\}$.

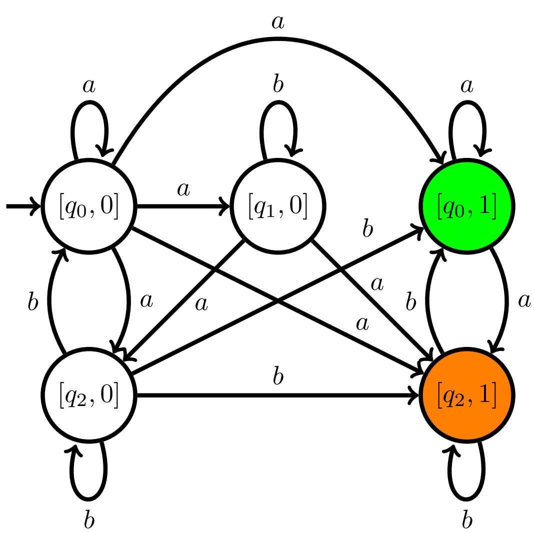
**NRA → NBA.** Given a Rabin automaton $A = \big(Q, \Sigma, Q_0, \delta, \{\langle F_0, G_0\rangle, \ldots, \langle F_{m-1}, G_{m-1}\rangle\}\big)$, it follows easily that, as in the case of Muller automata, $L_\omega(A) = \bigcup_{i=0}^{m-1} L_\omega(A_i)$ holds for the NRAs $A_i = (Q, \Sigma, Q_0, \delta, \{\langle F_i, G_i\rangle\})$. So it suffices to translate each $A_i$ into an NBA. Since an accepting run $\rho$ of $A_i$ satisfies $\inf(\rho) \cap G_i = \emptyset$, from some point on $\rho$ only visits states of $Q_i \setminus G_i$. So $\rho$ consists of an initial *finite* part, say $\rho_0$, that may visit all states, and an infinite part, say $\rho_1$, that only visits states of $Q \setminus G_i$. So we take two copies of $A_i$. Intuitively, $A_i'$ simulates $\rho$ by executing $\rho_0$ in the first copy, and $\rho_1$ in the second. The condition that $\rho_1$ must visit some state of $F_i$ infinitely often is enforced by taking $F_i$ as Büchi condition.

**Solution 12.4**

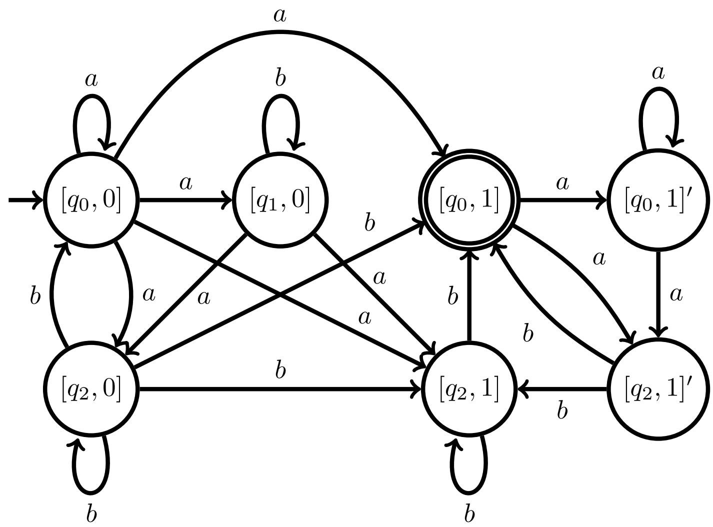
(a) We must first construct two generalized Büchi automata $A$ and $B$ for $\{q_1\}$ and $\{q_0, q_2\}$ respectively. Automaton $A$ is as follows with acceptance condition $\{\{q_1\}\}$:



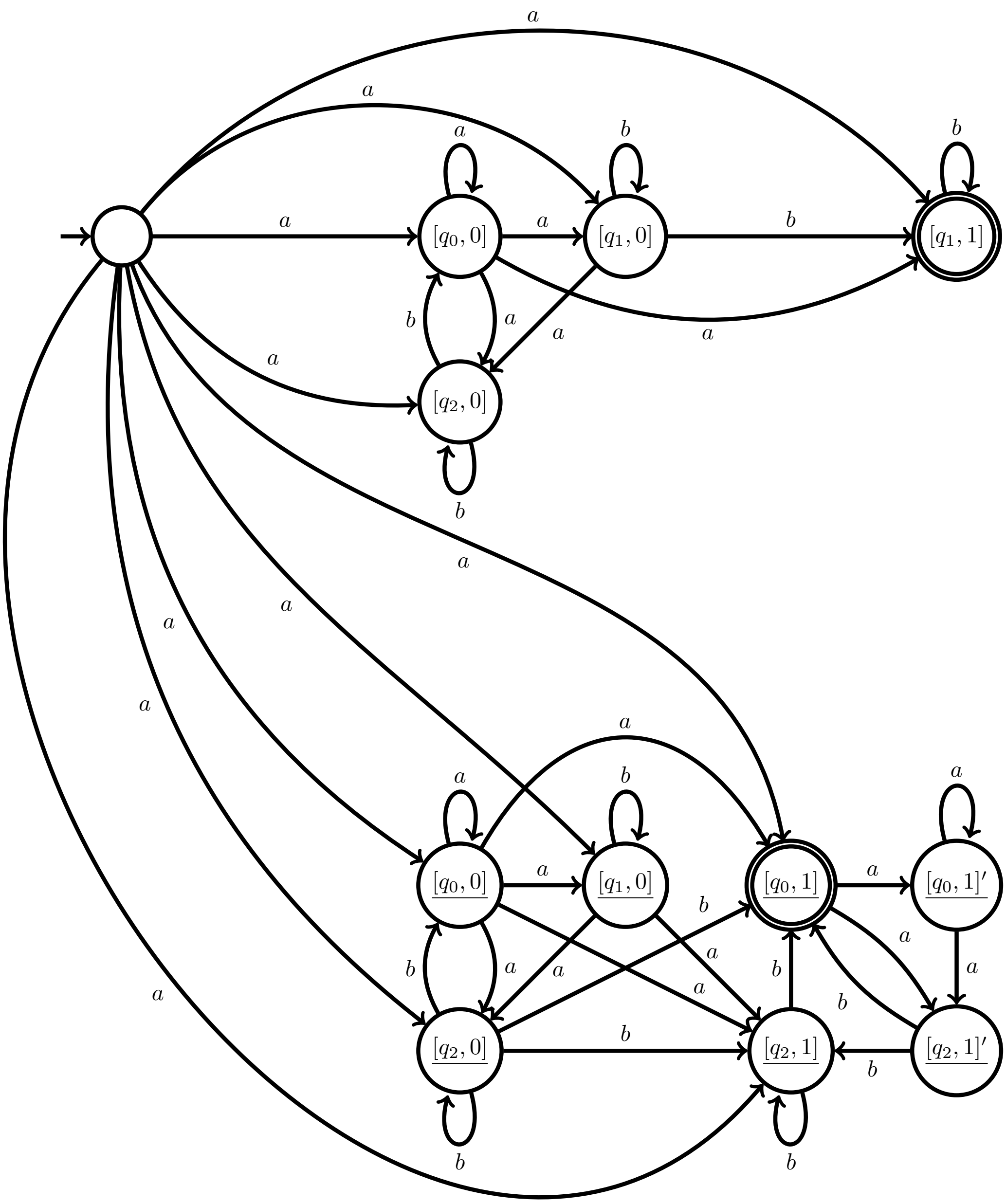

Automaton $B$ is as follows with acceptance condition $\{\{q_0\}, \{q_2\}\}$:




The resulting generalized Büchi automaton is the union of $A$ and $B$. Note that $A$ is essentially already a standard Büchi automaton, it suffices to make state $[q_1, 1]$ accepting. However, it remains to convert $B$ into a standard Büchi automaton $B'$:

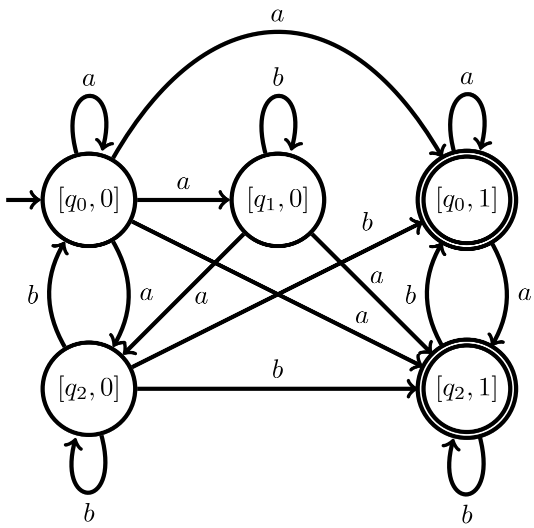



Altogether, we obtain the following Büchi automaton:




★ Since Büchi automata can have multiple initial states, we can also simply take the disjoint union of both automata, i.e. have them side by side instead of adding a single new initial.

(b)



a
a
b
a
[q_0, 0]
a
[q_1, 0]
[q_0, 1]
b
a
b
a
a
b
a
a
b
[q_2, 0]
[q_2, 1]
b
b