Automata and Formal Languages — Exercise Sheet 11

Exercise 11.1
Consider the logic PureMSO(Σ) with syntax

$$\varphi := X \subseteq Q_a \mid X < Y \mid X \subseteq Y \mid \neg\varphi \mid \varphi \vee \varphi \mid \exists X.\ \varphi$$

Notice that formulas of PureMSO(Σ) do not contain first-order variables. The satisfaction relation of PureMSO(Σ) is given by:

$$\begin{array}{lcll} (w, \mathcal{J}) & \models & X \subseteq Q_a & \text{iff} \quad w[p] = a \text{ for every } p \in \mathcal{J}(X) \\ (w, \mathcal{J}) & \models & X < Y & \text{iff} \quad p < p' \text{ for every } p \in \mathcal{J}(X),\ p' \in \mathcal{J}(Y) \\ (w, \mathcal{J}) & \models & X \subseteq Y & \text{iff} \quad p \in \mathcal{J}(Y) \text{ for every } p \in \mathcal{J}(X) \end{array}$$

with the rest as for MSO(Σ).

Prove that MSO(Σ) and PureMSO(Σ) have the same expressive power for sentences. That is, show that for every sentence ϕ of MSO(Σ) there is an equivalent sentence ψ of PureMSO(Σ), and vice versa.

Exercise 11.2
Give *deterministic* Büchi automata recognizing the following ω-languages over $\Sigma = \{a, b, c\}$:

(a) $L_1 = \{w \in \Sigma^\omega : w \text{ contains at least one } c\}$,

(b) $L_2 = \{w \in \Sigma^\omega : \text{in } w\text{, every } a \text{ is immediately followed by a } b\}$,

(c) $L_3 = \{w \in \Sigma^\omega : \text{in } w\text{, between two successive } a\text{'s there are at least two } b\text{'s}\}$.

Exercise 11.3
Let $\inf(w)$ denote the set of letters occurring infinitely often in the infinite word w. Give Büchi automata and ω-regular expressions for the following ω-languages over $\Sigma = \{a, b, c\}$:

(a) $L_1 = \{w \in \Sigma^\omega : \inf(w) \subseteq \{a, b\}\}$,

(b) $L_2 = \{w \in \Sigma^\omega : \inf(w) = \{a, b\}\}$,

(c) $L_3 = \{w \in \Sigma^\omega : \{a, b\} \subseteq \inf(w)\}$,

(d) $L_4 = \{w \in \Sigma^\omega : \inf(w) = \{a, b, c\}\}$.

(e) ★ Does there exist a deterministic Büchi automaton accepting L_1? If there is then give it, otherwise give a proof of why it is not true.

Solution 11.1
Given a sentence ψ of PureMSO(Σ), let ϕ be the sentence of MSO(Σ) obtained by replacing every subformula of ψ of the form

$$\begin{array}{lll} X \subseteq Y & \text{by} & \forall x\ (x \in X \rightarrow x \in Y) \\ X \subseteq Q_a & \text{by} & \forall x\ (x \in X \rightarrow Q_a(x)) \\ X < Y & \text{by} & \forall x\ \forall y\ (x \in X \wedge y \in Y) \rightarrow x < y \end{array}$$

Clearly, ϕ and ψ are equivalent. For the other direction, let

$$\text{empty}(X) := \forall Y\ X \subseteq Y$$

and

$$\text{sing}(X) := \neg\text{empty}(X) \wedge \forall Y\ (Y \subseteq X \wedge \neg\text{empty}(Y)) \rightarrow X = Y.$$

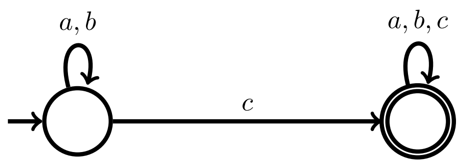
Let ϕ be a sentence of MSO(Σ). Assume without loss of generality that for every first-order variable x the second-order variable X does not appear in ϕ (if necessary, rename second-order variables appropiately). Let ψ be the sentence of PureMSO(Σ) obtained by replacing every subformula of ϕ of the form

$$\begin{array}{lll} \exists x\ \psi' & \text{by} & \exists X\ (\text{sing}(X) \wedge \psi'[x/X]) \\ & & \text{where } \psi'[x/X] \text{ is the result of substituting } X \text{ for } x \text{ in } \psi' \\ Q_a(x) & \text{by} & X \subseteq Q_a \\ x < y & \text{by} & X < Y \\ x \in Y & \text{by} & X \subseteq Y \end{array}$$

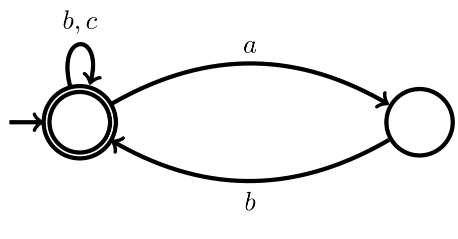
Clearly, ϕ and ψ are equivalent.

Solution 11.2

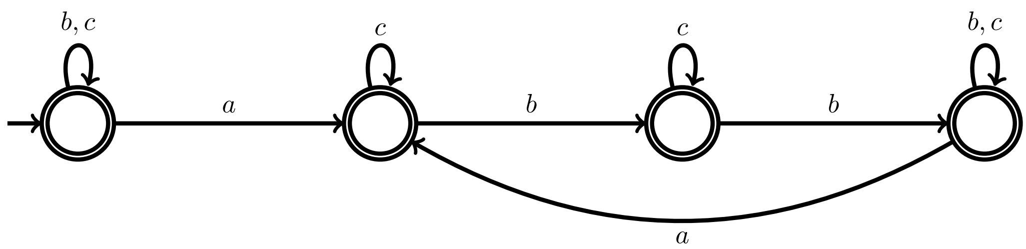
(a)



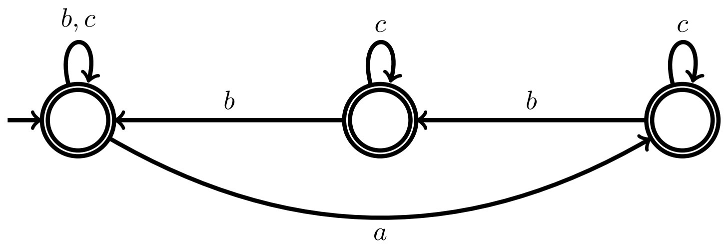
(b)



(c)

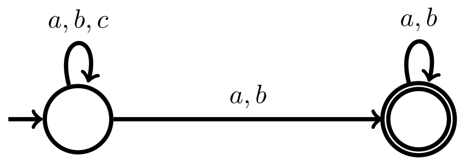


or simply,

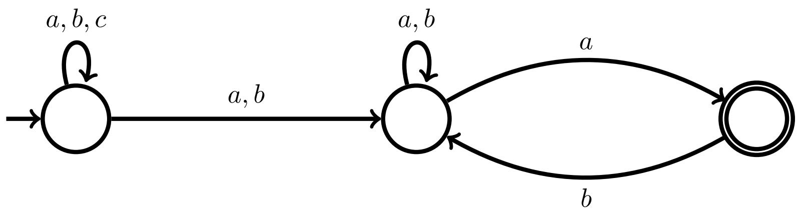


Solution 11.3

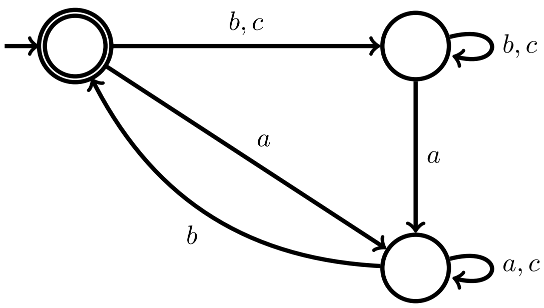
(a) $(a+b+c)^*(a+b)^\omega$, and



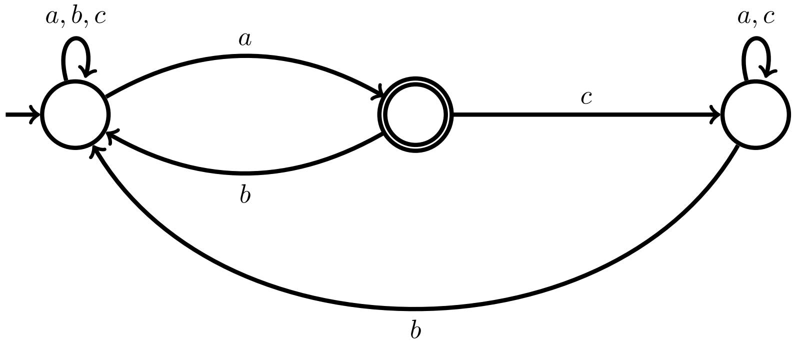
(b) $(a+b+c)^*(aa^*bb^*)^\omega$, and



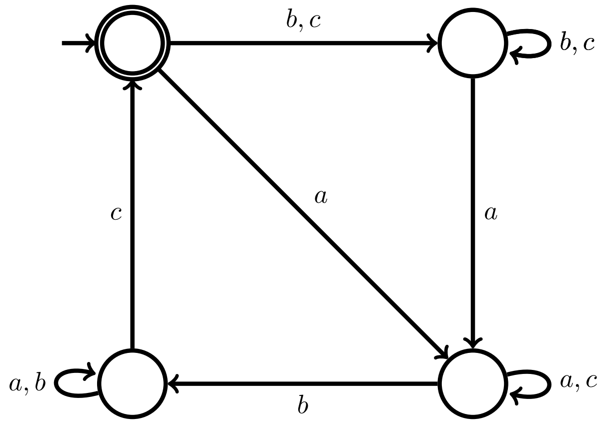
(c) $((b+c)^*a(a+c)^*b)^\omega$, and



or



(d) $((b+c)^*a(a+c)^*b(a+b)^*c)^\omega$, and



(e) ★ It is asked whether there exists a deterministic Büchi automaton accepting L_1. We show that it is *not* the case. For the sake of contradiction, suppose there exists a deterministic Büchi automaton $B = (Q, \Sigma, \delta, q_0, F)$ such that $L_\omega(B) = L_1$. Since $cb^\omega \in L_1$, B must visit F infinitely often when reading cb^ω. In particular, this implies the existence of $m_1 > 0$ and $q_1 \in F$ such that $q_0 \xrightarrow{cb^{m_1}} q_1$. Similarly, since $b^{m_1}cb^\omega \in L_1$, there exist $m_2 > 0$ and $q_2 \in F$ such that $q_0 \xrightarrow{cb^{m_1}cb^{m_2}} q_2$. Since B is deterministic, we have $q_0 \xrightarrow{cb^{m_1}} q_1 \xrightarrow{cb^{m_2}} q_2$. By repeating this argument $|Q|$ times, we can construct $m_1, m_2, \ldots, m_{|Q|} > 0$ and $q_1, q_2, \ldots, q_{|Q|} \in F$ such that

$$q_0 \xrightarrow{cb^{m_1}} q_1 \xrightarrow{cb^{m_2}} q_2 \cdots \xrightarrow{cb^{m_{|Q|}}} q_{|Q|}.$$

By the pigeonhole principle, there exist $0 \leq i < j \leq |Q|$ such that $q_i = q_j$. Let

$$u = cb^{m_1}cb^{m_2} \cdots cb^{m_i},$$
$$v = cb^{m_{i+1}}cb^{m_{i+2}} \cdots cb^{m_j}.$$

We have $q_0 \xrightarrow{u} q_i \xrightarrow{v} q_i \xrightarrow{v} q_i \xrightarrow{u} \cdots$ which implies that $uv^\omega \in L_\omega(B)$. This is a contradiction since $uv^\omega \notin L_1$. □