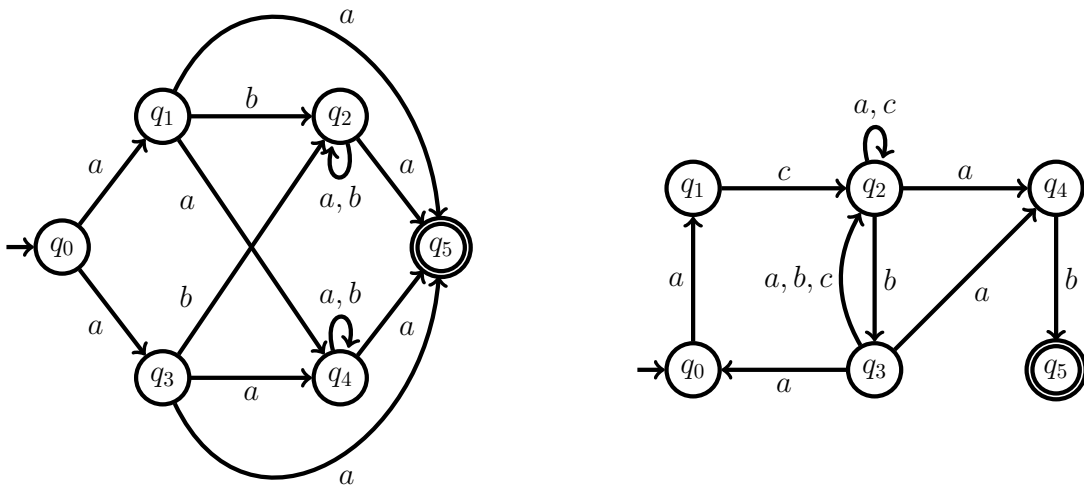


## Automata and Formal Languages — Exercise Sheet 4

**Exercise 4.1**

Let  $A$  and  $B$  be respectively the following NFAs:



- (a) Compute the coarsest stable refinements (CSR) of  $A$  and  $B$ . This is computed by the algorithm for reducing NFAs presented in the lecture.
- (b) Construct the quotients of  $A$  and  $B$  with respect to their CSRs.
- (c) Show that

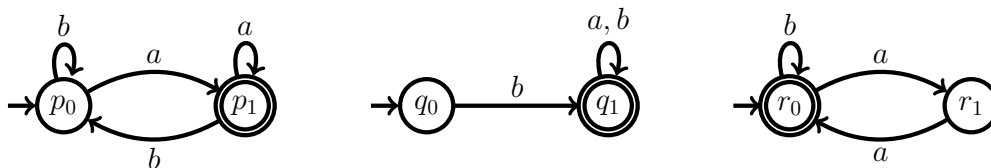
$$L(A) = \{w \in \{a,b\}^* \mid w \text{ starts and ends with } a\}$$

$$L(B) = \{w \in \{a,b\}^* \mid w \text{ starts with } ac \text{ and ends with } ab\}$$

- (d) Are the automata obtained in (b) minimal?

**Exercise 4.2**

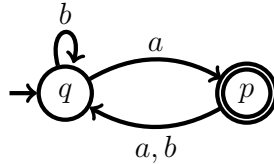
Consider the following DFAs  $A$ ,  $B$  and  $C$ :



Use pairings to decide *algorithmically* whether  $L(A) \cap L(B) \subseteq L(C)$ .

### Exercise 4.3

Let  $L \subseteq \Sigma^*$  be a language accepted by an NFA  $A$ . For every  $u, v \in \Sigma^*$ , we say that  $u \preceq v$  if and only if  $u$  can be obtained by deleting zero, one or multiple letters of  $v$ . For example,  $abc \preceq abca$ ,  $abc \preceq acbac$ ,  $abc \preceq abc$ ,  $\varepsilon \preceq abc$  and  $aab \not\preceq acbac$ . Consider the following NFA  $A$ . Give an NFA- $\varepsilon$  for each of the following languages and then generalize your approach to any NFA:



- (a)  $\downarrow L = \{w \in \Sigma^* \mid w \preceq w' \text{ for some } w' \in L\}$ ,
- (b)  $\uparrow L = \{w \in \Sigma^* \mid w' \preceq w \text{ for some } w' \in L\}$ ,
- (c)  $\sqrt{L} = \{w \in \Sigma^* \mid ww \in L\}$ ,
- (d)  $\star \text{Cyc}(L) = \{vu \in \Sigma^* \mid uv \in L\}$ .

### Exercise 4.4

Let  $\Sigma_1$  and  $\Sigma_2$  be alphabets. A *morphism* is a function  $h : \Sigma_1^* \rightarrow \Sigma_2^*$  such that  $h(\varepsilon) = \varepsilon$  and  $h(uv) = h(u) \cdot h(v)$  for every  $u, v \in \Sigma_1^*$ . In particular,  $h(a_1 a_2 \cdots a_n) = h(a_1) h(a_2) \cdots h(a_n)$  for every  $a_1, a_2, \dots, a_n \in \Sigma$ . Hence, a morphism  $h$  is entirely determined by its image over letters.

- (a) Let  $L \subseteq \Sigma_1^*$  be accepted by some NFA  $A_1$ . Give an NFA- $\varepsilon$   $B_2$  that accepts  $h(L) = \{h(w) \mid w \in L\}$ .
- (b) Show that  $L = \{(aab)^n e^m (cad)^n e f (fe)^n \mid m, n \in \mathbb{N}\}$  is not regular by using (a) and the fact that  $\{a^n b^n \mid n \in \mathbb{N}\}$  is not regular.

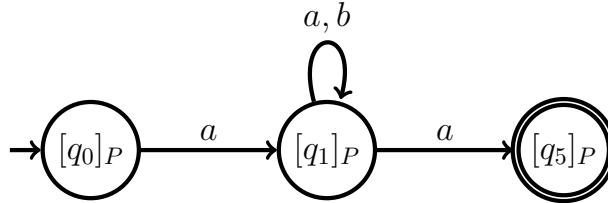
**Solution 4.1**

A) (a)

Iter.	Block to split	Splitter	New partition
0	—	—	$\{q_0, q_1, q_2, q_3, q_4\}, \{q_5\}$
1	$\{q_0, q_1, q_2, q_3, q_4\}$	$(a, \{q_5\})$	$\{q_0\}, \{q_1, q_2, q_3, q_4\}, \{q_5\}$
2	none, partition is stable	—	—

The CSR is  $P = \{\{q_0\}, \{q_1, q_2, q_3, q_4\}, \{q_5\}\}$ .

(b)



(c) It follows immediately from the fact that  $A$  accepts the same language as the automaton obtained in (b).

(d) Yes. By (c), the language accepted by  $A$  is  $a(a + b)^*a$ . An NFA with one state can only accept  $\emptyset, \{\varepsilon\}, a^*, b^*$  and  $\{a, b\}^*$ . Suppose there exists an NFA  $A' = (\{q_0, q_1\}, \{a, b\}, \delta, Q_0, F)$  accepting  $L(A)$ . Without loss of generality, we may assume that  $q_0$  is initial.  $A'$  must respect the following properties:

- $q_0 \notin F$ , since  $\varepsilon \notin L(A)$ ,
- $q_1 \in F$ , since  $L(A) \neq \emptyset$ ,
- $q_1 \notin Q_0$ , since  $\varepsilon \notin L(A)$ ,
- $q_1 \in \delta(q_0, a)$ , otherwise it is impossible to accept  $aa$  which is in  $L(A)$ .

This implies that  $A'$  accepts  $a$ , yet  $a \notin L(A)$ . Therefore, no two states NFA accepts  $L(A)$ .  $\square$

B) (a)

Iter.	Block to split	Splitter	New partition
0	—	—	$\{q_0, q_1, q_2, q_3, q_4\}, \{q_5\}$
1	$\{q_0, q_1, q_2, q_3, q_4\}$	$(b, \{q_5\})$	$\{q_0, q_1, q_2, q_3\}, \{q_4\}, \{q_5\}$
2	$\{q_0, q_1, q_2, q_3\}$	$(a, \{q_4\})$	$\{q_0, q_1\}, \{q_2, q_3\}, \{q_4\}, \{q_5\}$
3	$\{q_0, q_1\}$	$(c, \{q_2, q_3\})$	$\{q_0\}, \{q_1\}, \{q_2, q_3\}, \{q_4\}, \{q_5\}$
4	$\{q_2, q_3\}$	$(a, \{q_0\})$	$\{q_0\}, \{q_1\}, \{q_2\}, \{q_3\}, \{q_4\}, \{q_5\}$

The CSR is  $P = \{\{q_0\}, \{q_1\}, \{q_2\}, \{q_3\}, \{q_4\}, \{q_5\}\}$ .

(b) The automaton remains unchanged.

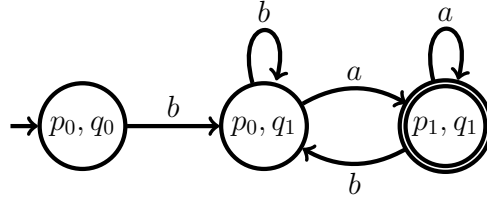
(c)  $\subseteq$ ) Let  $w \in L(B)$ . Every path from  $q_0$  to  $q_5$  first goes through  $q_1$  and  $q_2$  and ends up going through  $q_4$  and  $q_5$ . This implies that  $w \in L(ac(a + b + c)^*ab)$ .

$\supseteq$ ) First note that for every  $u \in \{a, b, c\}^*$ , there exists  $q \in \{q_2, q_3\}$  such that  $q_2 \xrightarrow{u} q$ . This can be shown by induction on  $|u|$ . Let  $w \in L(ac(a + b + c)^*ab)$ . There exists  $u \in \{a, b, c\}^*$  such that  $w = acuab$ . Let  $q \in \{q_2, q_3\}$  be such that  $q_2 \xrightarrow{u} q$ . We have  $q_0 \xrightarrow{a} q_1 \xrightarrow{c} q_2 \xrightarrow{u} q \xrightarrow{a} q_4 \xrightarrow{b} q_5$ . Therefore,  $w \in L(B)$ .  $\square$

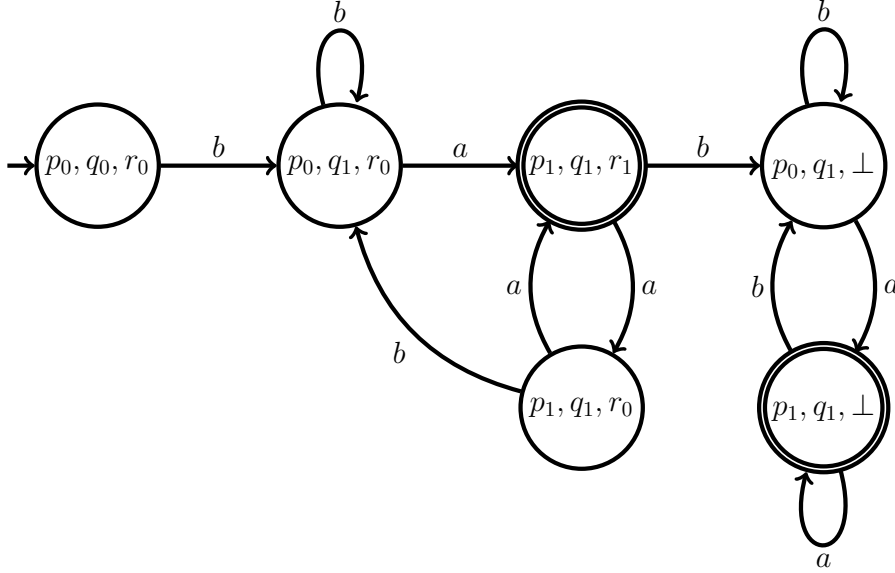
(d) No. We have seen a DFA with five states accepting the same language in Exercise #1.1.

**Solution 4.2**

We first build the pairing accepting  $L(A) \cap L(B)$ . Note that it is not necessary to explore the implicit trap states of  $A$  and  $B$  as they cannot lead to final states in the pairing. We obtain:



Now, we build the pairing accepting  $(L(A) \cap L(B)) \setminus L(C)$  from the above automaton and  $C$ . Note that we must explore the implicit trap state of  $C$  as it may be part of final states in the pairing. We obtain:



Since the above automaton contains final states, its language is non empty and hence  $L(A) \cap L(B) \not\subseteq L(C)$ . Note that we can reach this conclusion as soon as we construct state  $(p_1, q_1, r_1)$ .

### Solution 4.3

Let  $A = (Q, \Sigma, \delta, Q_0, F)$  be an NFA that accepts  $L$ .

- (a) We add a  $\varepsilon$ -transition “parallel” to every transition of  $A$ . This simulates the deletion of letters from words of  $L$ . More formally, let  $B = (Q, \Sigma, \delta', Q_0, F)$  be such that, for every  $q \in Q$  and  $a \in \Sigma \cup \{\varepsilon\}$ ,

$$\delta'(q, a) = \begin{cases} \delta(q, a) & \text{if } a \in \Sigma, \\ \{q \in Q : q \in \delta(q, b) \text{ for some } b \in \Sigma\} & \text{if } a = \varepsilon. \end{cases}$$

- (b) For every state of  $Q$ , we add self-loops for each letter of  $\Sigma$ . This corresponds to the insertion of letters in words of  $L$ . More formally, let  $B = (Q, \Sigma, \delta', Q_0, F)$  be such that  $\delta'(q, a) = \delta(q, a) \cup \{q\}$  for every  $q \in Q$  and  $a \in \Sigma$ .
- (c) Intuitively, we construct an automaton  $B$  that guesses an intermediate state  $p$  and then reads  $w$  simultaneously from an initial state  $q_0$  and from  $p$ . The automaton accepts if it simultaneously reaches  $p$  and an accepting state  $q_F$ . More formally, let  $B = (Q', \Sigma, \delta', Q'_0, F')$  be such that

$$\begin{aligned} Q' &= Q \times Q \times Q, \\ Q'_0 &= \{(p, q, p) : p \in Q, q \in Q_0\}, \\ F' &= \{(p, p, q) : p \in Q, q \in F\}, \end{aligned}$$

and, for every  $p, q, r \in Q$  and  $a \in \Sigma$ ,

$$\delta'((p, q, r), a) = \{(p, q', r') : q' \in \delta(q, a), r' \in \delta(r, a)\}.$$

- (d) Intuitively, we construct an automaton  $B$  that guesses a state  $p$  and reads a prefix  $v$  of the input word until it reaches a final state. Then,  $B$  moves non deterministically to an initial state from which it reads the remainder  $u$  of the input word, and it accepts if it reaches  $p$ . More formally, let  $B = (Q', \Sigma, \delta', Q'_0, F')$  be such that

$$\begin{aligned} Q' &= Q \times \{0, 1\} \times Q, \\ Q'_0 &= \{(p, 0, p) \mid p \in Q\}, \\ F' &= \{(p, 1, p) \mid p \in Q\}, \end{aligned}$$

and, for every  $p, q \in Q$  and  $a \in \Sigma \cup \{\varepsilon\}$ ,

$$\delta'((p, b, q), a) = \begin{cases} \{(p, b, q') : q' \in \delta(q, a)\} & \text{if } a \in \Sigma, \\ \{(p, 1, q') : q' \in Q_0\} & \text{if } a = \varepsilon, b = 0 \text{ and } q \in F, \\ \emptyset & \text{otherwise.} \end{cases}$$

#### Solution 4.4

- (a) Since  $h$  is determined by its image over letters, we replace each transition  $(p, a, q)$  of  $A$  by a sequence of transitions from  $p$  to  $q$  labeled by  $h(a)$ . Some  $\varepsilon$ -transitions may be introduced if  $h(a) = \varepsilon$  for some  $a \in \Sigma$ .
- (b) Suppose  $L$  is regular and let  $h$  be the morphism such that  $h(b) = a$ ,  $h(c) = b$  and  $h(a) = h(d) = h(e) = h(f) = \varepsilon$ . We have

$$h(L) = \{a^n b^n : n \in \mathbb{N}\}.$$

By (a), language  $h(L)$  is regular, which is a contradiction. □