

Automata and Formal Languages — Homework 14

Due 05.02.2019

Exercise 14.1

Let $AP = \{p, q\}$ and let $\Sigma = 2^{AP}$. Give LTL formulas for the following ω -languages:

- (a) $\{p, q\} \emptyset \Sigma^\omega$
- (b) $\Sigma^* \{q\}^\omega$
- (c) $\Sigma^* (\{p\} + \{p, q\}) \Sigma^* \{q\} \Sigma^\omega$
- (d) $\{p\}^* \{q\}^* \emptyset^\omega$

Exercise 14.2

Let $AP = \{p, q\}$ and let $\Sigma = 2^{AP}$. Give Büchi automata for the ω -languages over Σ defined by the following LTL formulas:

- (a) $\mathbf{XG}\neg p$
- (b) $(\mathbf{GF}p) \rightarrow (\mathbf{F}q)$
- (c) $p \wedge \neg(\mathbf{XF}p)$
- (d) $\mathbf{G}(p \mathbf{U} (p \rightarrow q))$
- (e) $\mathbf{F}q \rightarrow (\neg q \mathbf{U} (\neg q \wedge p))$

Exercise 14.3

Let $A = (Q, \Sigma, \delta, q_0, F)$ be an automaton such that $Q = P \times [n]$ for some finite set P and $n \geq 1$. Automaton A models a system made of n processes. A state $(p, i) \in Q$ represents the current global state p of the system, and the last process i that was executed.

We define two predicates exec_j and enab_j over Q indicating whether process j is respectively executed and enabled. More formally, for every $q = (p, i) \in Q$ and $j \in [n]$, let

$$\begin{aligned}\text{exec}_j(q) &\iff i = j, \\ \text{enab}_j(q) &\iff (p, i) \rightarrow (p', j) \text{ for some } p' \in P.\end{aligned}$$

- (a) Give LTL formulas over Q^ω for the following statements:
 - (i) All processes are executed infinitely often.
 - (ii) If a process is enabled infinitely often, then it is executed infinitely often.
 - (iii) If a process is eventually permanently enabled, then it is executed infinitely often.
- (b) The three above properties are known respectively as *unconditional*, *strong* and *weak* fairness. Show the following implications, and show that the reverse implications do not hold:

$$\text{unconditional fairness} \implies \text{strong fairness} \implies \text{weak fairness}.$$

Exercise 14.4

Let $AP = \{p, q\}$ and let $\Sigma = 2^{AP}$. An LTL formula is a tautology if it is satisfied by all computations. Which of the following LTL formulas are tautologies?

(a) $\mathbf{G}p \rightarrow \mathbf{F}p$

(b) $\mathbf{G}(p \rightarrow q) \rightarrow (\mathbf{G}p \rightarrow \mathbf{G}q)$

(c) $\mathbf{F}\mathbf{G}p \vee \mathbf{F}\mathbf{G}\neg p$

(d) $\neg\mathbf{F}p \rightarrow \mathbf{F}\neg\mathbf{F}p$

(e) $(\mathbf{G}p \rightarrow \mathbf{F}q) \leftrightarrow (p \mathbf{U} (\neg p \vee q))$

(f) $\neg(p \mathbf{U} q) \leftrightarrow (\neg p \mathbf{U} \neg q)$

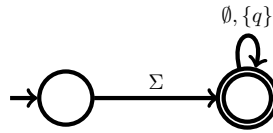
(g) $\mathbf{G}(p \rightarrow \mathbf{X}p) \rightarrow (p \rightarrow \mathbf{G}p)$

Solution 14.1

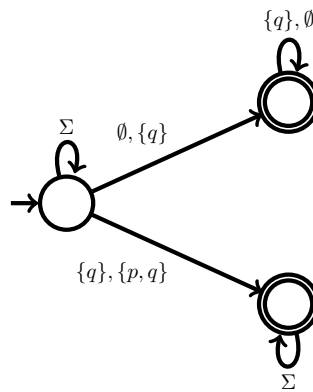
- (a) $(p \wedge q) \wedge \mathbf{X}(\neg p \wedge \neg q)$
- (b) $\mathbf{FG}(\neg p \wedge q)$
- (c) $\mathbf{F}(p \wedge \mathbf{XF}(\neg p \wedge q))$
- (d) $(p \wedge \neg q) \mathbf{U} ((\neg p \wedge q) \mathbf{U} \mathbf{G}(\neg p \wedge \neg q))$

Solution 14.2

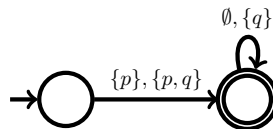
(a)



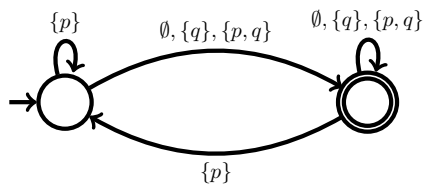
(b) Note that $(\mathbf{GF}p) \rightarrow (\mathbf{F}q) \equiv \neg(\mathbf{GF}p) \vee (\mathbf{F}q) \equiv (\mathbf{FG}\neg p) \vee (\mathbf{F}q)$. We construct Büchi automata for $\mathbf{FG}\neg p$ and $\mathbf{F}q$, and take their union:



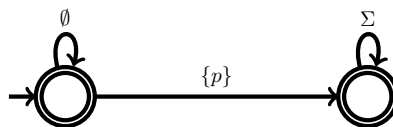
(c) Note that $p \wedge \neg(\mathbf{XF}p) \equiv p \wedge \mathbf{XG}\neg p$. We construct a Büchi automaton for $p \wedge \mathbf{XG}\neg p$:



(d)



(e)



Solution 14.3

- (a) (i) $\bigwedge_{j \in [n]} \mathbf{GF} \text{ exec}_j$
(ii) $\bigwedge_{j \in [n]} (\mathbf{GF} \text{ enab}_j \rightarrow \mathbf{GF} \text{ exec}_j)$
(iii) $\bigwedge_{j \in [n]} (\mathbf{FG} \text{ enab}_j \rightarrow \mathbf{GF} \text{ exec}_j)$
- (b) • Unconditional fairness implies strong fairness. For the sake of contradiction, suppose unconditional fairness holds for some execution σ , but not strong fairness. By assumption, there exists $j \in [n]$ such that $\sigma \not\models (\mathbf{GF} \text{ enab}_j \rightarrow \mathbf{GF} \text{ exec}_j)$. Thus,

$$\begin{aligned} \sigma &\not\models (\mathbf{GF} \text{ enab}_j \rightarrow \mathbf{GF} \text{ exec}_j) && \iff \\ \sigma &\models \neg(\mathbf{GF} \text{ enab}_j \rightarrow \mathbf{GF} \text{ exec}_j) && \iff \\ \sigma &\models \neg(\neg \mathbf{GF} \text{ enab}_j \vee \mathbf{GF} \text{ exec}_j) && \iff \\ \sigma &\models \mathbf{GF} \text{ enab}_j \wedge \neg \mathbf{GF} \text{ exec}_j && \implies \\ \sigma &\models \neg \mathbf{GF} \text{ exec}_j \end{aligned}$$

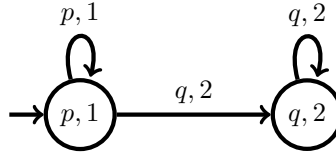
which contradicts unconditional fairness. □

- Strong fairness implies weak fairness. For the sake of contradiction, suppose strong fairness holds for some execution σ , but not weak fairness. By assumption, there exists $j \in [n]$ such that $\sigma \not\models (\mathbf{FG} \text{ enab}_j \rightarrow \mathbf{GF} \text{ exec}_j)$. Thus,

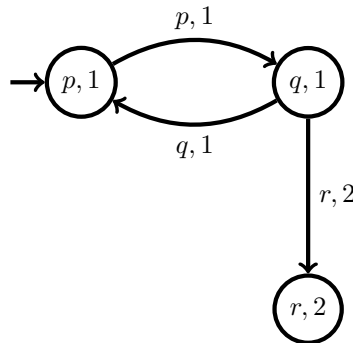
$$\begin{aligned} \sigma &\not\models (\mathbf{FG} \text{ enab}_j \rightarrow \mathbf{GF} \text{ exec}_j) && \iff \\ \sigma &\models \neg(\mathbf{FG} \text{ enab}_j \rightarrow \mathbf{GF} \text{ exec}_j) && \iff \\ \sigma &\models \neg(\neg \mathbf{FG} \text{ enab}_j \vee \mathbf{GF} \text{ exec}_j) && \iff \\ \sigma &\models \mathbf{FG} \text{ enab}_j \wedge \neg \mathbf{GF} \text{ exec}_j && \implies \\ \sigma &\models \mathbf{GF} \text{ enab}_j \wedge \neg \mathbf{GF} \text{ exec}_j && \iff \\ \sigma &\models \neg(\mathbf{GF} \text{ enab}_j \rightarrow \mathbf{GF} \text{ exec}_j) && \iff \\ \sigma &\not\models \mathbf{GF} \text{ enab}_j \rightarrow \mathbf{GF} \text{ exec}_j \end{aligned}$$

which contradicts strong fairness. □

- Strong fairness does not imply unconditional fairness. Execution $(p, 1)(q, 2)^\omega$ of the automaton below satisfies strong fairness, but not unconditional fairness.



- Weak fairness does not imply strong fairness. Execution $((p, 1)(q, 1))^\omega$ of the automaton below satisfies weak fairness, but not strong fairness.



Solution 14.4

(a) $\mathbf{G}p \rightarrow \mathbf{F}p$ is a tautology since

$$\begin{aligned} \sigma \models \mathbf{G}p &\iff \forall k \geq 0 \sigma^k \models p \\ &\implies \exists k \geq 0 \sigma^k \models p \\ &\iff \sigma \models \mathbf{F}p. \end{aligned}$$

(b) $\mathbf{G}(p \rightarrow q) \rightarrow (\mathbf{G}p \rightarrow \mathbf{G}q)$ is a tautology. For the sake of contradiction, suppose this is not the case. There exists σ such that

$$\sigma \models \mathbf{G}(p \rightarrow q), \text{ and} \tag{1}$$

$$\sigma \not\models (\mathbf{G}p \rightarrow \mathbf{G}q). \tag{2}$$

By (??), we have

$$\sigma \models \mathbf{G}p, \text{ and}$$

$$\sigma \not\models \mathbf{G}q.$$

Therefore, there exists $k \geq 0$ such that $p \in \sigma(k)$ and $q \notin \sigma(k)$ which contradicts (??).

(c) $\mathbf{F}\mathbf{G}p \vee \mathbf{F}\mathbf{G}\neg p$ is not a tautology since it is not satisfied by $(\{p\}\{q\})^\omega$.

(d) $\neg\mathbf{F}p \rightarrow \mathbf{F}\neg\mathbf{F}p$ is a tautology since $\varphi \rightarrow \mathbf{F}\varphi$ is a tautology for every formula φ .

(e) $(\mathbf{G}p \rightarrow \mathbf{F}q) \leftrightarrow (p \mathbf{U} (\neg p \vee q))$ is a tautology. We have

$$\begin{aligned} \mathbf{G}p \rightarrow \mathbf{F}q &\equiv \neg\mathbf{G}p \vee \mathbf{F}q && \text{(by def. of implication)} \\ &\equiv \mathbf{F}\neg p \vee \mathbf{F}q \\ &\equiv \mathbf{F}(\neg p \vee q) \\ &\equiv \mathbf{F}(p \rightarrow q) && \text{(by def. of implication)} \end{aligned}$$

Therefore, we have to show that

$$\mathbf{F}(p \rightarrow q) \leftrightarrow (p \mathbf{U} (p \rightarrow q)).$$

\leftarrow) Let σ be such that $\sigma \models (p \mathbf{U} (p \rightarrow q))$. In particular, there exists $k \geq 0$ such that $\sigma^k \models (p \rightarrow q)$. Therefore, $\sigma \models \mathbf{F}(p \rightarrow q)$.

\rightarrow) Let σ be such that $\sigma \models \mathbf{F}(p \rightarrow q)$. Let $k \geq 0$ be the smallest position such that $\sigma^k \models (p \rightarrow q)$. For every $0 \leq i < k$, we have $\sigma^i \not\models (p \rightarrow q)$ which is equivalent to $\sigma^i \models p \wedge \neg q$. Therefore, for every $0 \leq i < k$, we have $\sigma^i \models p$. This implies that $\sigma \models p \mathbf{U} (p \rightarrow q)$.

(f) $\neg(p \mathbf{U} q) \leftrightarrow (\neg p \mathbf{U} \neg q)$ is not a tautology. Let $\sigma = \{p\}\{q\}^\omega$. We have $\sigma \not\models \neg(p \mathbf{U} q)$ and $\sigma \models (\neg p \mathbf{U} \neg q)$.

(g) $\mathbf{G}(p \rightarrow \mathbf{X}p) \rightarrow (p \rightarrow \mathbf{G}p)$ is a tautology since

$$\begin{aligned} \mathbf{G}(p \rightarrow \mathbf{X}p) \rightarrow (p \rightarrow \mathbf{G}p) &\equiv \neg\mathbf{G}(p \rightarrow \mathbf{X}p) \vee (p \rightarrow \mathbf{G}p) && \text{(by def. of implication)} \\ &\equiv \mathbf{F}(p \wedge \neg\mathbf{X}p) \vee \neg p \vee \mathbf{G}p \\ &\equiv \neg\mathbf{G}p \rightarrow (\neg p \vee (\mathbf{F}(p \wedge \mathbf{X}\neg p))) && \text{(by def. of implication)} \\ &\equiv \mathbf{F}\neg p \rightarrow (\neg p \vee (\mathbf{F}(p \wedge \mathbf{X}\neg p))) \\ &\equiv \mathbf{F}\neg p \rightarrow \mathbf{F}\neg p. \end{aligned}$$