Automata and Formal Languages — Homework 10

Due 08.01.2019

Exercise 10.1
Let $\Sigma = \{a, b\}$. Give formulations in plain English of the languages described by the following formulas of FO(Σ), and give a corresponding regular expression:

(a) $\exists x.\ \mathit{first}(x)$

(b) $\forall x.\ \mathit{first}(x)$

(c) $\neg\exists x.\exists y.\ \big(x < y \wedge Q_a(x) \wedge Q_b(y)\big) \wedge \forall x.\ \big(Q_b(x) \rightarrow \exists y.\ x < y \wedge Q_a(y)\big) \wedge \exists x.\neg\exists y.\ x < y$

Exercise 10.2
Let $\Sigma = \{a, b\}$. Give an MSO(Σ) sentence for the following languages:

(a) The set of words with an a at every odd position.

(b) The set of words with an even number of occurrences of a's.

(c) The set of words of odd length with an even number of occurrences of a's.

Exercise 10.3
Give a MSO(Σ) formula $\mathsf{Odd\_Card}(X)$ expressing that the cardinality of the set of positions X is odd.

Exercise 10.4
Consider the logic PureMSO(Σ) with syntax

$$\varphi := X \subseteq Q_a \mid X < Y \mid X \subseteq Y \mid \neg\varphi \mid \varphi \vee \varphi \mid \exists X.\ \varphi$$

Notice that formulas of PureMSO(Σ) do not contain first-order variables. The satisfaction relation of PureMSO(Σ) is given by:

$$\begin{array}{lcll} (w, \mathcal{J}) & \models & X \subseteq Q_a & \text{iff} \quad w[p] = a \text{ for every } p \in \mathcal{J}(X) \\ (w, \mathcal{J}) & \models & X < Y & \text{iff} \quad p < p' \text{ for every } p \in \mathcal{J}(X),\ p' \in \mathcal{J}(Y) \\ (w, \mathcal{J}) & \models & X \subseteq Y & \text{iff} \quad p \in \mathcal{J}(Y) \text{ for every } p \in \mathcal{J}(X) \end{array}$$

with the rest as for MSO(Σ).

Prove that MSO(Σ) and PureMSO(Σ) have the same expressive power for sentences. That is, show that for every sentence ϕ of MSO(Σ) there is an equivalent sentence ψ of PureMSO(Σ), and vice versa.

Exercise 10.5
Let Σ be a finite alphabet. A language $L \subseteq \Sigma^*$ is *star-free* if it can be expressed by a star-free regular expression, i.e. a regular expression where the Kleene star operation is forbidden, but complementation is allowed. For example, Σ^* is star-free since $\Sigma^* = \overline{\emptyset}$, but $(aa)^*$ is not.

(a) Give star-free regular expressions and FO(Σ) sentences for the following star-free languages:

(i) Σ^+.

(ii) $\Sigma^* A \Sigma^*$ for some $A \subseteq \Sigma$.

(iii) A^* for some $A \subseteq \Sigma$.

(iv) $(ab)^*$.

(v) $\{w \in \Sigma^* \mid w \text{ does not contain } aa\ \}$.

(b) Show that finite and cofinite languages are star-free.

(c) Show that for every sentence $\varphi \in \mathrm{FO}(\Sigma)$, there exists a formula $\varphi^+(x, y)$, with two free variables x and y, such that for every $w \in \Sigma^+$ and for every $1 \le i \le j \le w$,

$$w \models \varphi^+(i, j) \quad \text{iff} \quad w_i w_{i+1} \cdots w_j \models \varphi \ .$$

(d) Give a polynomial time algorithm that decides whether the empty word satisfies a given sentence of $\mathrm{FO}(\Sigma)$.

(e) Show that every star-free language can be expressed by an $\mathrm{FO}(\Sigma)$ sentence.