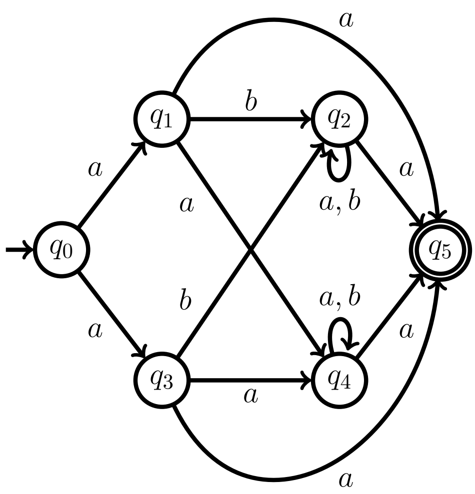
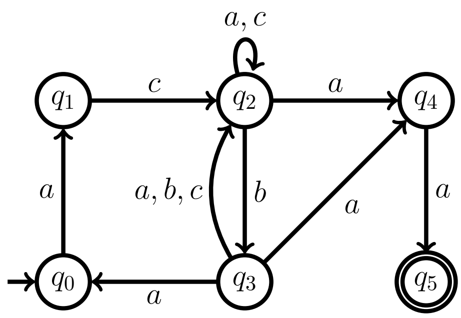
Technische Universität München **Winter term 2018/19**
I7
Prof. J. Esparza / S. Sickert **8.11.2018**

Automata and Formal Languages — Homework 4

Due 12.11.2018

Exercise 4.1
Let A and B be respectively the following NFAs:





(a) Compute the coarsest stable refinements (CSR) of A and B. This is computed by the algorithm for reducing NFAs presented in the lecture.

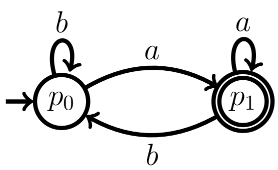
(b) Construct the quotients of A and B with respect to their CSRs.

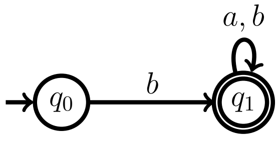
(c) Show that

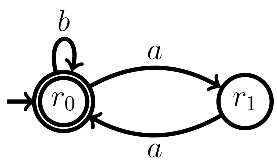
$$L(A) = \{w \in \{a,b\}^* \mid w \text{ starts and ends with } a\}$$
$$L(B) = \{w \in \{a,b\}^* \mid w \text{ starts with } ac \text{ and ends with } ab\}$$

(d) Are the automata obtained in (b) minimal?

Exercise 4.2
Consider the following DFAs A, B and C:



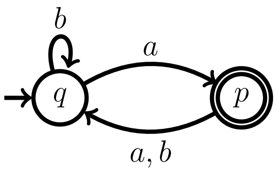




Use pairings to decide *algorithmically* whether $L(A) \cap L(B) \subseteq L(C)$.

Exercise 4.3

Let $L \subseteq \Sigma^*$ be a language accepted by an NFA A. For every $u, v \in \Sigma^*$, we say that $u \preceq v$ if and only if u can be obtained by deleting zero, one or multiple letters of v. For example, $abc \preceq abca$, $abc \preceq acbac$, $abc \preceq abc$, $\varepsilon \preceq abc$ and $aab \not\preceq acbac$. Consider the following NFA A. Give an NFA-ε for each of the following languages and then generalize your approach to any NFA:



(a) $\downarrow L = \{w \in \Sigma^* \mid w \preceq w' \text{ for some } w' \in L\}$,

(b) $\uparrow L = \{w \in \Sigma^* \mid w' \preceq w \text{ for some } w' \in L\}$,

(c) $\sqrt{L} = \{w \in \Sigma^* \mid ww \in L\}$,

(d) $\text{Cyc}(L) = \{vu \in \Sigma^* \mid uv \in L\}$.

Exercise 4.4

Let Σ_1 and Σ_2 be alphabets. A *morphism* is a function $h : \Sigma_1^* \to \Sigma_2^*$ such that $h(\varepsilon) = \varepsilon$ and $h(uv) = h(u) \cdot h(v)$ for every $u, v \in \Sigma_1^*$. In particular, $h(a_1 a_2 \cdots a_n) = h(a_1)h(a_2) \cdots h(a_n)$ for every $a_1, a_2, \ldots, a_n \in \Sigma$. Hence, a morphism h is entirely determined by its image over letters.

(a) Let $L \subseteq \Sigma_1^*$ be accepted by some NFA A_1. Give an NFA-ε B_2 that accepts $h(L) = \{h(w) \mid w \in L\}$.

(b) Show that $L = \{(aab)^n e^m (cad)^n ef(fe)^n \mid m, n \in \mathbb{N}\}$ is not regular by using (a) and the fact that $\{a^n b^n \mid n \in \mathbb{N}\}$ is not regular.

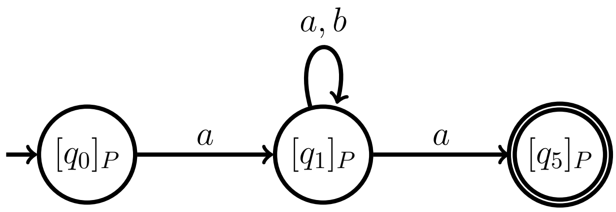
Solution 4.1

A) (a)

Iter.	Block to split	Splitter	New partition
0	—	—	$\{q_0, q_1, q_2, q_3, q_4\}, \{q_5\}$
1	$\{q_0, q_1, q_2, q_3, q_4\}$	$(a, \{q_5\})$	$\{q_0\}, \{q_1, q_2, q_3, q_4\}, \{q_5\}$
2	none, partition is stable	—	—

The CSR is $P = \{\{q_0\}, \{q_1, q_2, q_3, q_4\}, \{q_5\}\}$.

(b)



(c) It follows immediately from the fact that A accepts the same language as the automaton obtained in (b).

(d) Yes. By (c), the language accepted by A is $a(a+b)^*a$. An NFA with one state can only accept $\emptyset, \{\varepsilon\}, a^*, b^*$ and $\{a, b\}^*$. Suppose there exists an NFA $A' = (\{q_0, q_1\}, \{a, b\}, \delta, Q_0, F)$ accepting $L(A)$. Without loss of generality, we may assume that q_0 is initial. A' must respect the following properties:

- $q_0 \notin F$, since $\varepsilon \notin L(A)$,
- $q_1 \in F$, since $L(A) \neq \emptyset$,
- $q_1 \notin Q_0$, since $\varepsilon \notin L(A)$,
- $q_1 \in \delta(q_0, a)$, otherwise it is impossible to accept aa which is in $L(A)$.

This implies that A' accepts a, yet $a \notin L(A)$. Therefore, no two states NFA accepts $L(A)$. $\square$

B) (a)

Iter.	Block to split	Splitter	New partition
0	—	—	$\{q_0, q_1, q_2, q_3, q_4\}, \{q_5\}$
1	$\{q_0, q_1, q_2, q_3, q_4\}$	$(a, \{q_5\})$	$\{q_0, q_1, q_2, q_3\}, \{q_4\}, \{q_5\}$
2	$\{q_0, q_1, q_2, q_3\}$	$(a, \{q_4\})$	$\{q_0, q_1\}, \{q_2, q_3\}, \{q_4\}, \{q_5\}$
3	$\{q_0, q_1\}$	$(c, \{q_2, q_3\})$	$\{q_0\}, \{q_1\}, \{q_2, q_3\}, \{q_4\}, \{q_5\}$
4	$\{q_2, q_3\}$	$(a, \{q_0\})$	$\{q_0\}, \{q_1\}, \{q_2\}, \{q_3\}, \{q_4\}, \{q_5\}$

The CSR is $P = \{\{q_0\}, \{q_1\}, \{q_2\}, \{q_3\}, \{q_4\}, \{q_5\}\}$.

(b) The automaton remains unchanged.

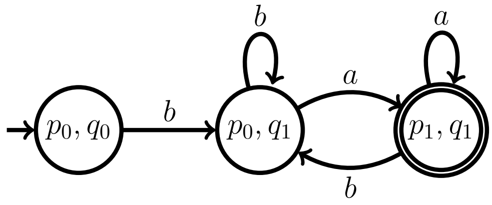
(c) $\subseteq$) Let $w \in L(B)$. Every path from q_0 to q_5 first goes through q_1 and q_2 and ends up going through q_4 and q_5. This implies that $w \in L(ac(a+b+c)^*ab)$.

$\supseteq$) First note that for every $u \in \{a, b, c\}^*$, there exists $q \in \{q_2, q_3\}$ such that $q_2 \xrightarrow{u} q$. This can be shown by induction on $|u|$. Let $w \in L(ac(a+b+c)^*ab)$. There exists $u \in \{a, b, c\}^*$ such that $w = acuab$. Let $q \in \{q_2, q_3\}$ be such that $q_2 \xrightarrow{u} q$. We have $q_0 \xrightarrow{a} q_1 \xrightarrow{c} q_2 \xrightarrow{u} q \xrightarrow{a} q_4 \xrightarrow{b} q_5$. Therefore, $w \in L(B)$. $\square$

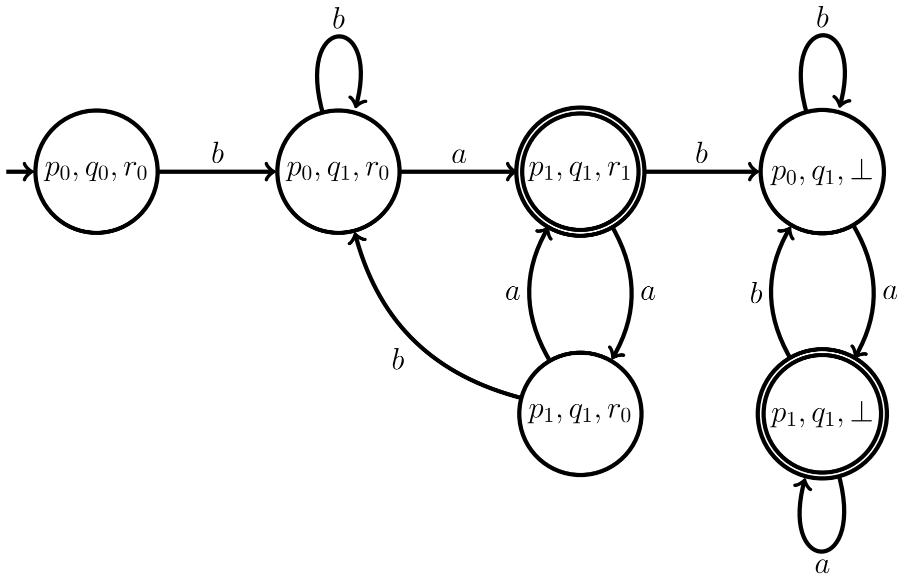
(d) No. We have seen a DFA with five states accepting the same language in Exercise #1.1.

Solution 4.2

We first build the pairing accepting $L(A) \cap L(B)$. Note that it is not necessary to explore the implicit trap states of A and B as they cannot lead to final states in the pairing. We obtain:



Now, we build the pairing accepting $(L(A) \cap L(B)) \setminus L(C)$ from the above automaton and C. Note that we must explore the implicit trap state of C as it may be part of final states in the pairing. We obtain:



Since the above automaton contains final states, its language is non empty and hence $L(A) \cap L(B) \not\subseteq L(C)$. Note that we can reach this conclusion as soon as we construct state (p_1, q_1, r_1).

Solution 4.3

Let $A = (Q, \Sigma, \delta, Q_0, F)$ be an NFA that accepts L.

(a) We add a ε-transition "parallel" to every transition of A. This simulates the deletion of letters from words of L. More formally, let $B = (Q, \Sigma, \delta', Q_0, F)$ be such that, for every $q \in Q$ and $a \in \Sigma \cup \{\varepsilon\}$,

$$\delta'(q, a) = \begin{cases} \delta(q, a) & \text{if } a \in \Sigma, \\ \{q \in Q : q \in \delta(q, b) \text{ for some } b \in \Sigma\} & \text{if } a = \varepsilon. \end{cases}$$

(b) For every state of Q, we add self-loops for each letter of Σ. This corresponds to the insertion of letters in words of L. More formally, let $B = (Q, \Sigma, \delta', Q_0, F)$ be such that $\delta'(q, a) = \delta(q, a) \cup \{q\}$ for every $q \in Q$ and $a \in \Sigma$.

(c) Intuitively, we construct an automaton B that guesses an intermediate state p and then reads w simultaneously from an initial state q_0 and from p. The automaton accepts if it simultaneously reaches p and and an accepting state q_F. More formally, let $B = (Q', \Sigma, \delta', Q'_0, F')$ be such that

$$\begin{aligned} Q' &= Q \times Q \times Q, \\ Q'_0 &= \{(p, q, p) : p \in Q, q \in Q_0\}, \\ F' &= \{(p, p, q) : p \in Q, q \in F\}, \end{aligned}$$

and, for every $p, q, r \in Q$ and $a \in \Sigma$,

$$\delta'((p, q, r), a) = \{(p, q', r') : q' \in \delta(q, a), r' \in \delta(r, a)\}.$$

(d) Intuitively, we construct an automaton B that guesses a state p and reads a prefix v of the input word until it reaches a final state. Then, B moves non deterministically to an initial state from which it reads the remainder u of the input word, and it accepts if it reaches p. More formally, let $B = (Q', \Sigma, \delta', Q'_0, F')$ be such that

$$\begin{aligned} Q' &= Q \times \{0,1\} \times Q, \\ Q'_0 &= \{(p, 0, p) \mid p \in Q\}, \\ F' &= \{(p, 1, p) \mid p \in Q\}, \end{aligned}$$

and, for every $p, q \in Q$ and $a \in \Sigma \cup \{\varepsilon\}$,

$$\delta'((p, b, q), a) = \begin{cases} \{(p, b, q') : q' \in \delta(q, a)\} & \text{if } a \in \Sigma, \\ \{(p, 1, q') : q' \in Q_0\} & \text{if } a = \varepsilon, b = 0 \text{ and } q \in F, \\ \emptyset & \text{otherwise.} \end{cases}$$

Solution 4.4

(a) Since h is determined by its image over letters, we replace each transition (p, a, q) of A by a sequence of transitions from p to q labeled by $h(a)$. Some ε-transitions may be introduced if $h(a) = \varepsilon$ for some $a \in \Sigma$.

(b) Let $A = (Q, \Sigma_2, \delta, Q_0, F)$. We keep the states of A unchanged, but we remove its transitions. For each $p, q \in Q$ and $a \in \Sigma_1$, we add a transition (p, a, q) to B for every state q that can be reached from state p by reading $h(a)$ in A. More formally, let $B = (Q, \Sigma_1, \delta', Q_0, F)$ be such that

$$\delta'(p, a) = \{q \in Q : p \xrightarrow{h(a)}_A q\}.$$

(c) For the sake of contradiction, suppose L is regular. There exists an NFA A that accepts L. Let g be the morphism such that $g(a) = a$, $g(b) = b$, $g(c) = c$, $g(d) = d$ and $g(e) = g(f) = \varepsilon$. We have

$$g(L) = \{(aab)^n (cad)^n : n \in \mathbb{N}\}.$$

By (a), language $g(L)$ is regular. Let h be the morphism such that $h(a) = aab$, $h(b) = cad$, $h(c) = c$, $h(d) = d$, $h(e) = e$ and $h(f) = f$. We have

$$h^{-1}(g(L)) = \{a^n b^n : n \in \mathbb{N}\}.$$

By (b), language $h^{-1}(g(L))$ is regular, which is a contradiction. □

★ As discussed in class, there is a simpler solution. Suppose L is regular and let h be the morphism such that $h(b) = a$, $h(c) = b$ and $h(a) = h(d) = h(e) = h(f) = \varepsilon$. We have

$$h(L) = \{a^n b^n : n \in \mathbb{N}\}.$$

By (a), language $h(L)$ is regular, which is a contradiction. □