Technische Universität München **Winter term 2017/18**
I7
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Automata and Formal Languages — Homework 14

Due 06.02.2018

Exercise 14.1
Let $\mathrm{AP} = \{p, q\}$ and let $\Sigma = 2^{\mathrm{AP}}$. Give LTL formulas for the following ω-languages:

(a) $\{p, q\}\, \emptyset\, \Sigma^\omega$

(b) $\Sigma^*\, \{q\}^\omega$

(c) $\Sigma^*\, (\{p\} + \{p, q\})\, \Sigma^*\, \{q\}\, \Sigma^\omega$

(d) $\{p\}^*\, \{q\}^*\, \emptyset^\omega$

Exercise 14.2
Let $\mathrm{AP} = \{p, q\}$ and let $\Sigma = 2^{\mathrm{AP}}$. Give Büchi automata for the ω-languages over Σ defined by the following LTL formulas:

(a) $\mathbf{XG}\neg p$

(b) $(\mathbf{GF}p) \rightarrow (\mathbf{F}q)$

(c) $p \wedge \neg(\mathbf{XF}p)$

(d) $\mathbf{G}(p\ \mathbf{U}\ (p \rightarrow q))$

(e) $\mathbf{F}q \rightarrow (\neg q\ \mathbf{U}\ (\neg q \wedge p))$

Exercise 14.3
Let $A = (Q, \Sigma, \delta, q_0, F)$ be an automaton such that $Q = P \times [n]$ for some finite set P and $n \geq 1$. Automaton A models a system made of n processes. A state $(p, i) \in Q$ represents the current global state p of the system, and the last process i that was executed.

We define two predicates exec_j and enab_j over Q indicating whether process j is respectively executed and enabled. More formally, for every $q = (p, i) \in Q$ and $j \in [n]$, let

$$\mathrm{exec}_j(q) \iff i = j,$$
$$\mathrm{enab}_j(q) \iff (p, i) \rightarrow (p', j) \text{ for some } p' \in P.$$

(a) Give LTL formulas over Q^ω for the following statements:
  (i) All processes are executed infinitely often.
  (ii) If a process is enabled infinitely often, then it is executed infinitely often.
  (iii) If a process is eventually permanently enabled, then it is executed infinitely often.

(b) The three above properties are known respectively as *unconditional*, *strong* and *weak* fairness. Show the following implications, and show that the reverse implications do not hold:

$$\text{unconditional fairness} \implies \text{strong fairness} \implies \text{weak fairness}.$$

Exercise 14.4

Let $\mathrm{AP} = \{p, q\}$ and let $\Sigma = 2^{\mathrm{AP}}$. An LTL formula is a tautology if it is satisfied by all computations. Which of the following LTL formulas are tautologies?

(a) $\mathbf{G}p \rightarrow \mathbf{F}p$

(b) $\mathbf{G}(p \rightarrow q) \rightarrow (\mathbf{G}p \rightarrow \mathbf{G}q)$

(c) $\mathbf{FG}p \vee \mathbf{FG}\neg p$

(d) $\neg\mathbf{F}p \rightarrow \mathbf{F}\neg\mathbf{F}p$

(e) $(\mathbf{G}p \rightarrow \mathbf{F}q) \leftrightarrow (p\ \mathbf{U}\ (\neg p \vee q))$

(f) $\neg(p\ \mathbf{U}\ q) \leftrightarrow (\neg p\ \mathbf{U}\ \neg q)$

(g) $\mathbf{G}(p \rightarrow \mathbf{X}p) \rightarrow (p \rightarrow \mathbf{G}p)$

Solution 14.1

(a) $(p \land q) \land \mathbf{X}(\neg p \land \neg q)$

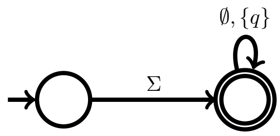
(b) $\mathbf{FG}(\neg p \land q)$

(c) $\mathbf{F}(p \land \mathbf{XF}(\neg p \land q))$

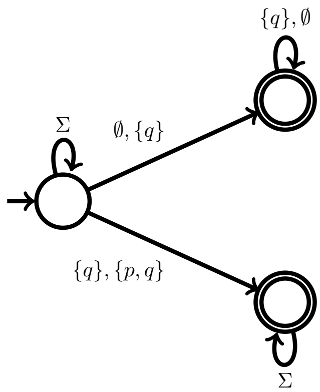
(d) $(p \land \neg q)\ \mathbf{U}\ ((\neg p \land q)\ \mathbf{U}\ \mathbf{G}(\neg p \land \neg q))$

Solution 14.2

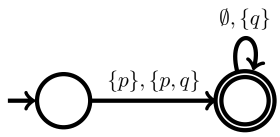
(a)



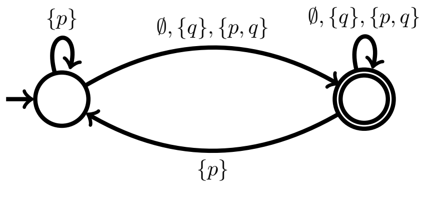
(b) Note that $(\mathbf{GF}p) \to (\mathbf{F}q) \equiv \neg(\mathbf{GF}p) \lor (\mathbf{F}q) \equiv (\mathbf{FG}\neg p) \lor (\mathbf{F}q)$. We construct Büchi automata for $\mathbf{FG}\neg p$ and $\mathbf{F}q$, and take their union:



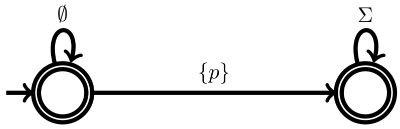
(c) Note that $p \land \neg(\mathbf{XF}p) \equiv p \land \mathbf{XG}\neg p$. We construct a Büchi automaton for $p \land \mathbf{XG}\neg p$:



(d)



(e)



Solution 14.3

(a) (i) $\bigwedge_{j \in [n]} \mathbf{GF}\ \text{exec}_j$

(ii) $\bigwedge_{j \in [n]} (\mathbf{GF}\ \text{enab}_j \to \mathbf{GF}\ \text{exec}_j)$

(iii) $\bigwedge_{j \in [n]} (\mathbf{FG}\ \text{enab}_j \to \mathbf{GF}\ \text{exec}_j)$

(b)

- Unconditional fairness implies strong fairness. For the sake of contradiction, suppose unconditional fairness holds for some execution σ, but not strong fairness. By assumption, there exists $j \in [n]$ such that $\sigma \not\models (\mathbf{GF}\ \text{enab}_j \to \mathbf{GF}\ \text{exec}_j)$. Thus,

$$\begin{array}{lll}
\sigma & \not\models & (\mathbf{GF}\ \text{enab}_j \to \mathbf{GF}\ \text{exec}_j) & \iff \\
\sigma & \models & \neg(\mathbf{GF}\ \text{enab}_j \to \mathbf{GF}\ \text{exec}_j) & \iff \\
\sigma & \models & \neg(\neg\mathbf{GF}\ \text{enab}_j \lor \mathbf{GF}\ \text{exec}_j) & \iff \\
\sigma & \models & \mathbf{GF}\ \text{enab}_j \land \neg\mathbf{GF}\ \text{exec}_j & \implies \\
\sigma & \models & \neg\mathbf{GF}\ \text{exec}_j &
\end{array}$$

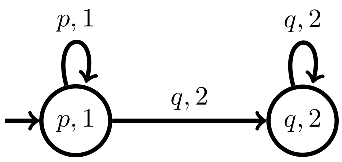
  which contradicts unconditional fairness. □

- Strong fairness implies weak fairness. For the sake of contradiction, suppose strong fairness holds for some execution σ, but not weak fairness. By assumption, there exists $j \in [n]$ such that $\sigma \not\models (\mathbf{FG}\ \text{enab}_j \to \mathbf{GF}\ \text{exec}_j)$. Thus,

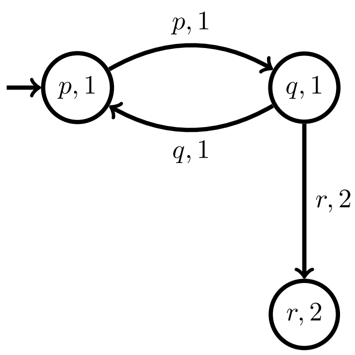
$$\begin{array}{lll}
\sigma & \not\models & (\mathbf{FG}\ \text{enab}_j \to \mathbf{GF}\ \text{exec}_j) & \iff \\
\sigma & \models & \neg(\mathbf{FG}\ \text{enab}_j \to \mathbf{GF}\ \text{exec}_j) & \iff \\
\sigma & \models & \neg(\neg\mathbf{FG}\ \text{enab}_j \lor \mathbf{GF}\ \text{exec}_j) & \iff \\
\sigma & \models & \mathbf{FG}\ \text{enab}_j \land \neg\mathbf{GF}\ \text{exec}_j & \implies \\
\sigma & \models & \mathbf{GF}\ \text{enab}_j \land \neg\mathbf{GF}\ \text{exec}_j & \iff \\
\sigma & \models & \neg(\mathbf{GF}\ \text{enab}_j \to \mathbf{GF}\ \text{exec}_j) & \iff \\
\sigma & \not\models & \mathbf{GF}\ \text{enab}_j \to \mathbf{GF}\ \text{exec}_j &
\end{array}$$

  which contradicts strong fairness. □

- Strong fairness does not imply unconditional fairness. Execution $(p, 1)(q, 2)^\omega$ of the automaton below satisfies strong fairness, but not unconditional fairness.



- Weak fairness does not imply strong fairness. Execution $((p, 1)(q, 1))^\omega$ of the automaton below satisfies weak fairness, but not strong fairness.



Solution 14.4

(a) $\mathbf{G}p \to \mathbf{F}p$ is a tautology since

$$\begin{aligned} \sigma \models \mathbf{G}p &\iff \forall k \geq 0\ \sigma^k \models p \\ &\implies \exists k \geq 0\ \sigma^k \models p \\ &\iff \sigma \models \mathbf{F}p. \end{aligned}$$

(b) $\mathbf{G}(p \to q) \to (\mathbf{G}p \to \mathbf{G}q)$ is a tautology. For the sake of contradiction, suppose this is not the case. There exists σ such that

$$\sigma \models \mathbf{G}(p \to q), \text{ and} \tag{1}$$
$$\sigma \not\models (\mathbf{G}p \to \mathbf{G}q). \tag{2}$$

By (**??**), we have

$$\begin{aligned} &\sigma \models \mathbf{G}p, \text{ and} \\ &\sigma \not\models \mathbf{G}q. \end{aligned}$$

Therefore, there exists $k \geq 0$ such that $p \in \sigma(k)$ and $q \notin \sigma(k)$ which contradicts (**??**).

(c) $\mathbf{FG}p \vee \mathbf{FG}\neg p$ is not a tautology since it is not satisfied by $(\{p\}\{q\})^\omega$.

(d) $\neg\mathbf{F}p \to \mathbf{F}\neg\mathbf{F}p$ is a tautology since $\varphi \to \mathbf{F}\varphi$ is a tautology for every formula φ.

(e) $(\mathbf{G}p \to \mathbf{F}q) \leftrightarrow (p\ \mathbf{U}\ (\neg p \vee q))$ is a tautology. We have

$$\begin{aligned} \mathbf{G}p \to \mathbf{F}q &\equiv \neg\mathbf{G}p \vee \mathbf{F}q && \text{(by def. of implication)} \\ &\equiv \mathbf{F}\neg p \vee \mathbf{F}q \\ &\equiv \mathbf{F}(\neg p \vee q) \\ &\equiv \mathbf{F}(p \to q) && \text{(by def. of implication)} \end{aligned}$$

Therefore, we have to show that

$$\mathbf{F}(p \to q) \leftrightarrow (p\ \mathbf{U}\ (p \to q)).$$

$\leftarrow$) Let σ be such that $\sigma \models (p\ \mathbf{U}\ (p \to q))$. In particular, there exists $k \geq 0$ such that $\sigma^k \models (p \to q)$. Therefore, $\sigma \models \mathbf{F}(p \to q)$.

$\to$) Let σ be such that $\sigma \models \mathbf{F}(p \to q)$. Let $k \geq 0$ be the smallest position such that $\sigma^k \models (p \to q)$. For every $0 \leq i < k$, we have $\sigma^i \not\models (p \to q)$ which is equivalent to $\sigma^i \models p \wedge \neg q$. Therefore, for every $0 \leq i < k$, we have $\sigma^i \models p$. This implies that $\sigma \models p\ \mathbf{U}\ (p \to q)$.

(f) $\neg(p\ \mathbf{U}\ q) \leftrightarrow (\neg p\ \mathbf{U}\ \neg q)$ is not a tautology. Let $\sigma = \{p\}\{q\}^\omega$. We have $\sigma \not\models \neg(p\ \mathbf{U}\ q)$ and $\sigma \not\models (\neg p\ \mathbf{U}\ \neg q)$.

(g) $\mathbf{G}(p \to \mathbf{X}p) \to (p \to \mathbf{G}p)$ is a tautology since

$$\begin{aligned} \mathbf{G}(p \to \mathbf{X}p) \to (p \to \mathbf{G}p) &\equiv \neg\mathbf{G}(\neg p \vee \mathbf{X}p) \vee (\neg p \vee \mathbf{G}p) && \text{(by def. of implication)} \\ &\equiv \mathbf{F}(p \wedge \neg\mathbf{X}p) \vee \neg p \vee \mathbf{G}p \\ &\equiv \neg\mathbf{G}p \to (\neg p \vee (\mathbf{F}(p \wedge \mathbf{X}\neg p)) && \text{(by def. of implication)} \\ &\equiv \mathbf{F}\neg p \to (\neg p \vee (\mathbf{F}(p \wedge \mathbf{X}\neg p)) \\ &\equiv \mathbf{F}\neg p \to \mathbf{F}\neg p. \end{aligned}$$