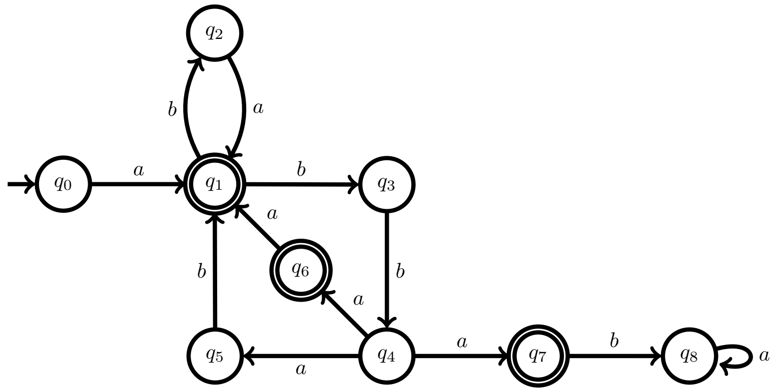
Automata and Formal Languages — Homework 13

Due 30.01.2018

Exercise 13.1
Let B be the following Büchi automaton:



(a) Execute the emptiness algorithm *NestedDFS* on B.

(b) Recall that *NestedDFS* is a non deterministic algorithm and different choices of runs may return different lassos. Which lassos of B can be found by *NestedDFS*?

(c) Show that *NestedDFS* is non optimal by exhibiting some search sequence on B.

(d) Execute the emptiness algorithm *TwoStack* on B.

(e) Which lassos of B can be found by *TwoStack*?

Exercise 13.2
Prove or disprove:

(a) $\mathbf{GF}(\varphi \vee \psi) \equiv \mathbf{GF}\varphi \vee \mathbf{GF}\psi$

(b) $\mathbf{GF}(\varphi \wedge \psi) \equiv \mathbf{GF}\varphi \wedge \mathbf{GF}\psi$

(c) $(\varphi \vee \psi)\ \mathbf{U}\ \rho \equiv (\varphi\ \mathbf{U}\ \rho) \vee (\psi\ \mathbf{U}\ \rho)$

(d) $\rho\ \mathbf{U}\ (\varphi \vee \psi) \equiv (\rho\ \mathbf{U}\ \varphi) \vee (\rho\ \mathbf{U}\ \psi)$

Exercise 13.3

Let $\mathrm{AP} = \{p, q, r\}$. Give formulas for the computations satisfying the following properties:

(a) if q eventually holds, then p may not hold before q first do

(b) if q eventually holds, then p holds before q first holds

(c) p always holds between q and r.

(d) p, and only p, holds at even positions and q, and only q, holds at odd positions.

Solution 13.1

(a) Let us assume that the algorithms always pick states in ascending order with respect to their indices. *dfs1* visits $q_0, q_1, q_2, q_3, q_4, q_5, q_6$, then calls *dfs2* which visits $q6, q1, q2, q3, q4, q5, q6$ and reports "non empty".

(b) Since q_7 does not belong to any lasso, only lassos containing q_1 or q_6 can be found. In every run of the algorithm, *dfs1* blackens q_6 before q_1. The only lasso containing q_6 is: $q_0, q_1, q_3, q_4, q_6, q_1$. Therefore, this is the only lasso that can be found by the algorithm.

(c) The execution given in (a) shows that *NestedDFS* is non optimal since it returns the lasso $q_0, q_1, q_3, q_4, q_6, q_1$ even though the lasso q_0, q_1, q_2, q_1 was already appearing in the explored subgraph.

(d) Let us assume that the algorithms always pick states in ascending order with respect to their indices. The algorithm reports "non empty" after the following execution:

$\xrightarrow{C.\text{push}(q_0),\ V.\text{push}(q_0)}$

C	V
q_0	q_0

$\xrightarrow{C.\text{push}(q_1),\ V.\text{push}(q_1)}$

C	V
q_1	q_1
q_0	q_0

$\xrightarrow{C.\text{push}(q_2),\ V.\text{push}(q_2)}$

C	V
q_2	q_2
q_1	q_1
q_0	q_0

$\xrightarrow{C.\text{pop}()}$

C	V
	q_2
q_1	q_1
q_0	q_0

$\xrightarrow{C.\text{pop}()}$

C	V
	q_2
	q_1
q_0	q_0

(e) All of them. The lasso q_0, q_1, q_2, q_1 is found by the above execution. The lasso $q_0, q_1, q_3, q_4, q_6, q_1$ is found by the following execution:

$\xrightarrow{C.\text{push}(q_0),\ V.\text{push}(q_0)}$

C	V
q_0	q_0

$\xrightarrow{C.\text{push}(q_1),\ V.\text{push}(q_1)}$

C	V
q_1	q_1
q_0	q_0

$\xrightarrow{C.\text{push}(q_3),\ V.\text{push}(q_3)}$

C	V
q_3	q_3
q_1	q_1
q_0	q_0

$\xrightarrow{C.\text{push}(q_4),\ V.\text{push}(q_4)}$

C	V
q_4	q_4
q_3	q_3
q_1	q_1
q_0	q_0

$\xrightarrow{C.\text{push}(q_6),\ V.\text{push}(q_6)}$

C	V
q_6	q_6
q_4	q_4
q_3	q_3
q_1	q_1
q_0	q_0

$\xrightarrow{C.\text{pop}()}$

C	V
	q_6
q_4	q_4
q_3	q_3
q_1	q_1
q_0	q_0

The lasso $q_0, q_1, q_3, q_4, q_5, q_1$ is found by the following execution:

$\xrightarrow{C.\text{push}(q_0),\ V.\text{push}(q_0)}$

C	V
q_0	q_0

$\xrightarrow{C.\text{push}(q_1),\ V.\text{push}(q_1)}$

C	V
q_1	q_1
q_0	q_0

$\xrightarrow{C.\text{push}(q_3),\ V.\text{push}(q_3)}$

C	V
q_3	q_3
q_1	q_1
q_0	q_0

$\xrightarrow{C.\text{push}(q_4),\ V.\text{push}(q_4)}$

C	V
q_4	q_4
q_3	q_3
q_1	q_1
q_0	q_0

$\xrightarrow{C.\text{push}(q_5),\ V.\text{push}(q_5)}$

C	V
q_5	q_5
q_4	q_4
q_3	q_3
q_1	q_1
q_0	q_0

$\xrightarrow{C.\text{pop}()}$

C	V
	q_5
q_4	q_4
q_3	q_3
q_1	q_1
q_0	q_0

$\xrightarrow{C.\text{pop}()}$

C	V
	q_5
	q_4
q_3	q_3
q_1	q_1
q_0	q_0

$\xrightarrow{C.\text{pop}()}$

C	V
	q_5
	q_4
	q_3
q_1	q_1
q_0	q_0

$\xrightarrow{C.\text{pop}()}$

C	V
	q_5
	q_4
	q_3
	q_1
q_0	q_0

Solution 13.2

(a) True. If $\sigma \models \mathbf{GF}\varphi \lor \mathbf{GF}\psi$, then $\sigma \models \mathbf{GF}(\varphi \lor \psi)$. If $\sigma \models \mathbf{GF}(\varphi \lor \psi)$, then there exist $i_0 < i_1 < \cdots$ such that

$$\sigma^{i_j} \models \varphi \lor \psi \text{ for every } j \in \mathbb{N}. \tag{1}$$

Let $I = \{j \in \mathbb{N} : \sigma^{i_j} \models \varphi\}$ and $J = \{j \in \mathbb{N} : \sigma^{i_j} \models \psi\}$. If I and J are both finite, then (1) does not hold, which is a contradiction. Therefore, at least one of I and J is infinite. This implies that $\sigma \models \mathbf{GF}\varphi \lor \mathbf{GF}\psi$. $\square$

(b) False. Let $\sigma = (\{p\}\{q\})^\omega$. We have $\sigma \not\models \mathbf{GF}(p \land q)$ and $\sigma \models \mathbf{GF}p \land \mathbf{GF}q$.

(c) False. Let $\sigma = \{p\}\{q\}\{r\}\emptyset^\omega$. We have $\sigma \models (p \lor q)\ \mathbf{U}\ r$ and $\sigma \not\models (p\ \mathbf{U}\ r) \lor (q\ \mathbf{U}\ r)$.

(d) True, since:

$$
\begin{aligned}
\sigma \models \rho \,\mathbf{U}\, (\varphi \vee \psi) &\iff \exists k \geq 0 \text{ s.t. } \sigma^k \models (\varphi \vee \psi) \wedge \forall 0 \leq i < k \ \sigma^i \models \rho \\
&\iff \exists k \geq 0 \text{ s.t. } ((\sigma^k \models \varphi) \vee (\sigma^k \models \psi)) \wedge \forall 0 \leq i < k \ \sigma^i \models \rho \\
&\iff \exists k \geq 0 \text{ s.t. } (\sigma^k \models \varphi \wedge \forall 0 \leq i < k \ \sigma^i \models \rho) \vee (\sigma^k \models \psi \wedge \forall 0 \leq i < k \ \sigma^i \models \rho) \\
&\iff (\exists k \geq 0 \text{ s.t. } \sigma^k \models \varphi \wedge \forall 0 \leq i < k \ \sigma^i \models \rho) \vee (\exists k \geq 0 \text{ s.t. } \sigma^k \models \psi \text{ and } \forall 0 \leq i < k \ \sigma^i \models \rho) \\
&\iff \sigma \models (\rho \,\mathbf{U}\, \varphi) \vee (\rho \,\mathbf{U}\, \psi). \qquad \square
\end{aligned}
$$

Solution 13.3

(a) $\mathbf{F}q \rightarrow (\neg p \,\mathbf{U}\, q)$

(b) $\mathbf{F}q \rightarrow (\neg q \,\mathbf{U}\, (\neg q \wedge p))$

(c) $\mathbf{G}((q \wedge \mathbf{XF}r) \rightarrow \mathbf{X}(p \,\mathbf{U}\, r))$

(d) $\mathbf{G}(\neg r) \wedge \mathbf{G}(p \leftrightarrow \neg q) \wedge p \wedge \mathbf{G}(p \rightarrow \mathbf{X}q) \wedge \mathbf{G}(q \rightarrow \mathbf{X}p)$