Technische Universität München **Winter term 2017/18**
I7
Prof. J. Esparza / Dr. M. Blondin **27.10.2017**

Automata and Formal Languages — Homework 2

Due 06.11.2017

Exercise 2.1
Consider the regular expression $r = (a + ab)^*$.

(a) Convert r into an equivalent NFA-ε A.

(b) Convert A into an equivalent NFA B. (It is not necessary to use algorithm *NFAεtoNFA*)

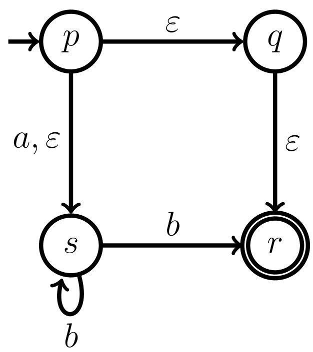
(c) Convert B into an equivalent DFA C.

(d) By inspecting B, give an equivalent minimal DFA D. (No algorithm needed).

(e) Convert D into an equivalent regular expression r'.

(f) Prove formally that $L(r) = L(r')$.

Exercise 2.2
Convert the following NFA-ε to an NFA using the algorithm *NFAεtoNFA* from the lecture notes (see Sect. 2.3, p. 33). You may verify your answer with the Python program `nfa-eps2nfa`.



Exercise 2.3
For every $n \in \mathbb{N}$, let $L_n = \{w \in \{0,1\}^* : |w| \geq n \text{ and } w_{|w|-n+1} = 1\}$.

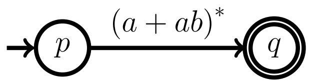
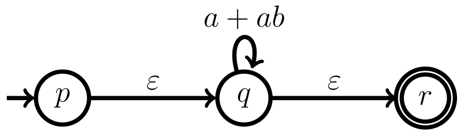
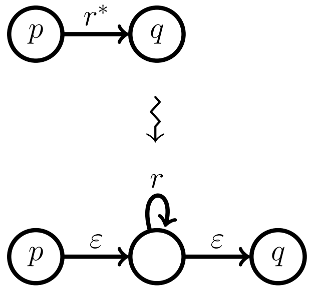
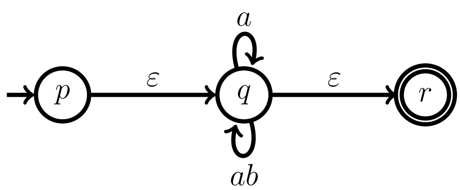
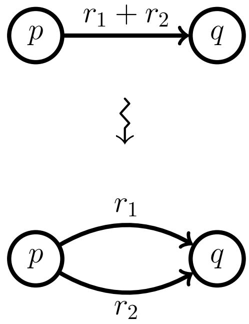
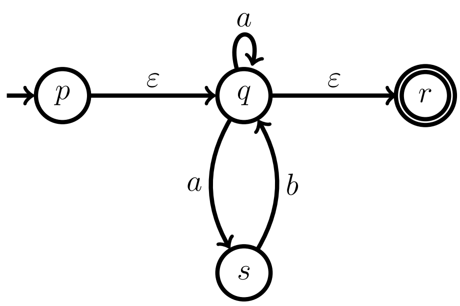
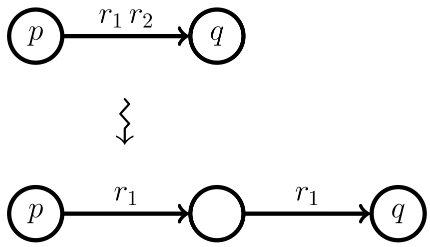
(a) Exhibit an NFA with $O(n)$ states that accepts L_n.

(b) Exhibit a DFA with $\Omega(2^n)$ states that accepts L_n.

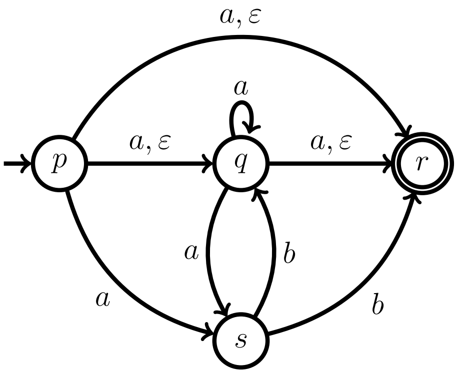
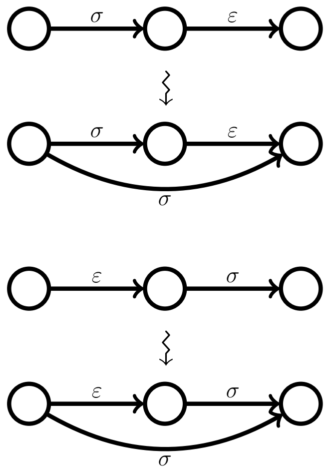
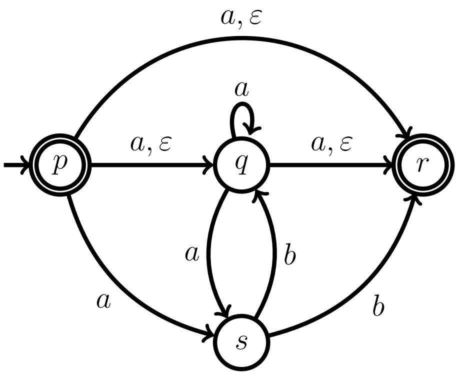
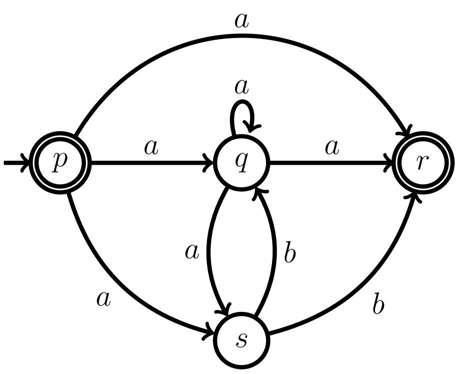
(c) Show that any DFA that accepts L_n has at least 2^n states.

Solution 2.1

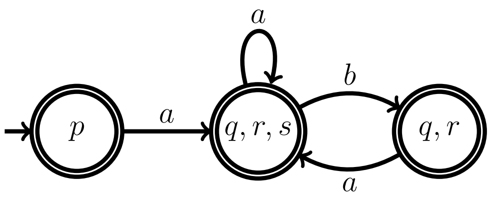
(a)

Iter.	Automaton obtained	Rule applied
1		Initial automaton from reg. expr.
2		
3		
4		

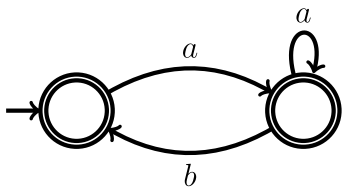
(b)

Iter.	Automaton obtained	Rule applied
1		 where $\sigma \in \Sigma \cup \{\varepsilon\}$
2		Initial states that can reach a final state through ε-transitions are made final.
3		Remove ε-transitions. Remove states non reachable from initial state.

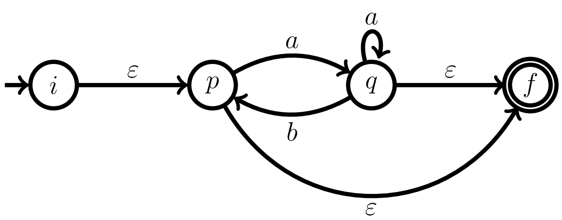
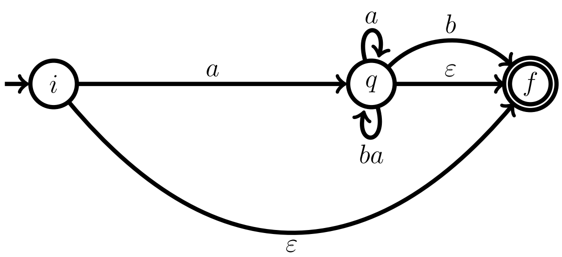
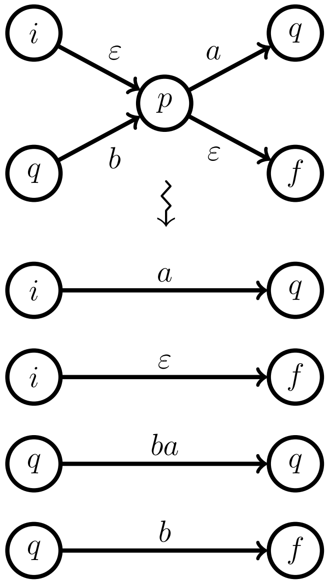
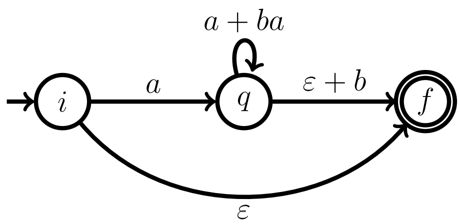
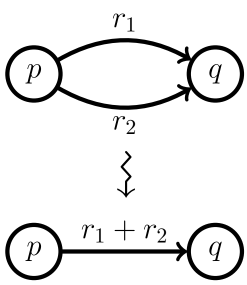
(c)

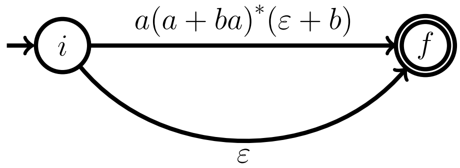
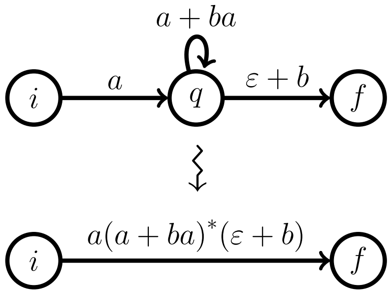
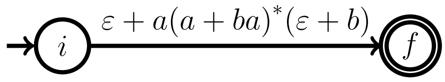
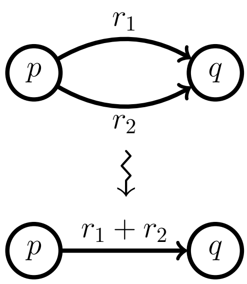


(d) States $\{p\}$ and $\{q,r\}$ have the exact same behaviours, so we can merge them. Indeed, both states are final and $\delta(\{p\},\sigma) = \delta(\{q,r\}),\sigma)$ for every $\sigma \in \{a,b\}$. We obtain:



(e)

Iter.	**Automaton obtained**	**Rule applied**
1		Add single initial and final states.
2		
3		

4		
5		
6	$\varepsilon + a(a+ba)^*(\varepsilon+b)$	Extract regular expression from the unique transition.

(f) Let us first show that $a(a+ba)^i = (a+ab)^i a$ for every $i \in \mathbb{N}$. We proceed by induction on i. If $i = 0$, then the claim trivially holds. Let $i > 0$. Assume the claims holds at $i-1$. We have

$$\begin{aligned}
a(a+ba)^i &= a(a+ba)^{i-1}(a+ba) \\
&= (a+ab)^{i-1}a(a+ba) && \text{(by induction hypothesis)} \\
&= (a+ab)^{i-1}(aa+aba) && \text{(by distributivity)} \\
&= (a+ab)^{i-1}(a+ab)a && \text{(by distributivity)} \\
&= (a+ab)^i a.
\end{aligned}$$

This implies that

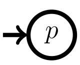
$$a(a+ba)^* = (a+ab)^* a. \tag{1}$$

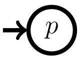
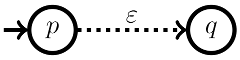
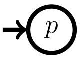
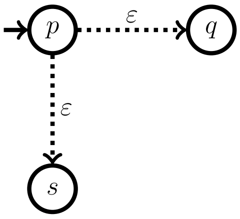
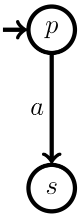
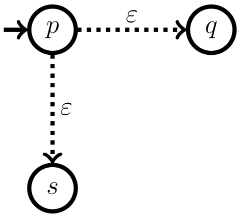
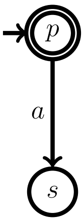
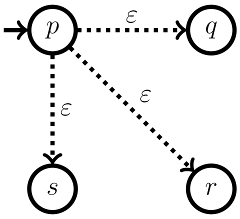
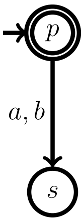
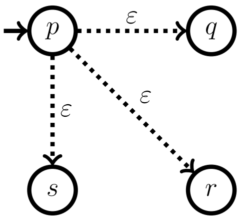
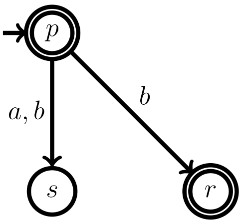
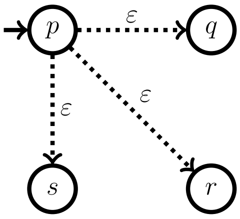
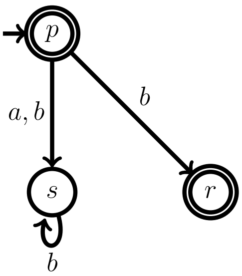
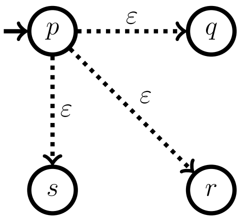
We may now prove the equivalence of the two regular expressions:

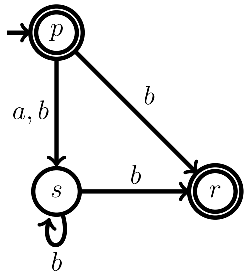
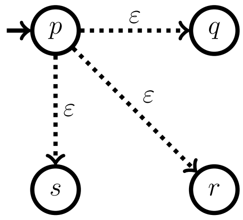
$$\begin{aligned}
\varepsilon + a(a+ba)^*(\varepsilon+b) &= \varepsilon + (a+ab)^* a(\varepsilon+b) && \text{(by (1))} \\
&= \varepsilon + (a+ab)^*(a+ab) && \text{(by distributivity)} \\
&= \varepsilon + (a+ab)^+ \\
&= (a+ab)^*.
\end{aligned}$$

□

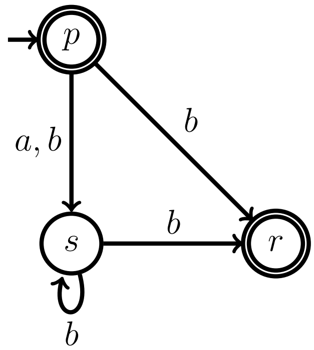
Solution 2.2

Iter.	$B = (Q', \Sigma, \delta', Q'_0, F')$	δ'' (**ε-transitions**)	**Workset** W **and next** (q_1, α, q_2)
0			$\{(p,\varepsilon,q), (p,\varepsilon,s), (p,a,s)\}$

1			$\{(p, \varepsilon, s), (p, a, s), (p, \varepsilon, r)\}$
2			$\{(p, a, s), (p, \varepsilon, r), (p, b, s), (p, b, r)\}$
3			$\{(p, \varepsilon, r), (p, b, s), (p, b, r), (s, b, s), (s, b, r)\}$
4			$\{(p, b, s), (p, b, r), (s, b, s), (s, b, r)\}$
5			$\{(p, b, r), (s, b, s), (s, b, r)\}$
6			$\{(s, b, s), (s, b, r)\}$
7			$\{(s, b, r)\}$

<table>
<tr><td>8</td><td>
</td><td>
</td><td>∅</td></tr>
</table>

The resulting NFA is:

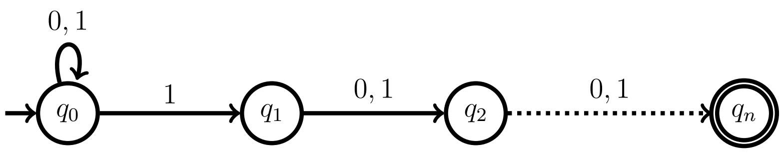


which corresponds to the output of `nfa-eps2nfa`:

```
Q'  = {'p', 'r', 's'}
S   = {'a', 'b'}
d'  = {('p', 'a', 's'), ('s', 'b', 's'), ('p', 'b', 's'), ('s', 'b', 'r'), ('p', 'b', 'r')}
Q0' = {'p'}
F'  = {'p', 'r'}
```

Solution 2.3

(a)



(b) We build a DFA that remembers the last n letters and accepts if the n to last last letter is a 1. More formally, let $A_n = (Q, \Sigma, \delta, q_0, F)$ be such that

$$
\begin{aligned}
Q &= \{q_u : u \in \{0,1\}^*, |u| \leq n\},\\
\Sigma &= \{0,1\},\\
q_0 &= q_\varepsilon,\\
F &= \{q_{1u} : u \in \{0,1\}^*, |u| = n-1\},
\end{aligned}
$$

and such that

$$
\delta(q_u, a) = \begin{cases} q_{ua} & \text{if } |u| < n,\\ q_{va} & \text{if } u = bv \text{ for some } b \in \{0,1\} \text{ and } v \in \{0,1\}^{n-1}. \end{cases}
$$

Note that A_n has $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$ states.

(c) Let $n \in \mathbb{N}$. For the sake of contradiction, assume there exists a DFA $B = (Q, \{0,1\}, \delta, q_0, F)$ such that $L(B) = L_n$ and $|Q| < 2^n$. By the pigeonhole principle, there exist $u, v \in \{0,1\}^n$ and $q \in Q$ such that $u \neq v$ and

$$q_0 \xrightarrow{u} q \text{ and } q_0 \xrightarrow{v} q. \tag{2}$$

Since $u \neq v$, there exists $1 \leq i \leq n$ such that $u_i \neq v_i$. Without loss of generality, we may assume that $u_i = 1$ and $v_i = 0$. We have $u \cdot 0^{i-1} \in L_n$ and $v \cdot 0^{i-1} \notin L$. This is a contradiction since, by (2), $u \cdot 0^{i-1}$ and $v \cdot 0^{i-1}$ lead to the same state from q_0. $\square$